Department of Mathematics
Eighth Annual
High School Problem Solving Competition
November 1, 2023

Solutions

1. **10 points** Suppose that a, b, c, d, and e are real numbers such that

$$a+b+c+d+e=8$$

and

$$a^2+b^2+c^2+d^2+e^2=16.$$

Determine the maximum possible value of e.

Solution:

From Cauchy's inequality

$$(a\cdot 1+b\cdot 1+c\cdot 1+d\cdot 1)^2 \le (a^2+b^2+c^2+d^2)(1+1+1+1)$$

we deduce that

$$(8-e)^2 \le 4(16-e^2).$$

We simplify this inequality to $e(5e-16)\le 0$ and conclude that $0\le e\le 16/5$. Thus the maximum value of e is $16/5$ which is attained when $a=b=c=d=6/5$.

2. **10 points** Let q be an odd integer greater than 1. Show that there is a positive integer n such that 2^n-1 is a multiple of q.

Solution:

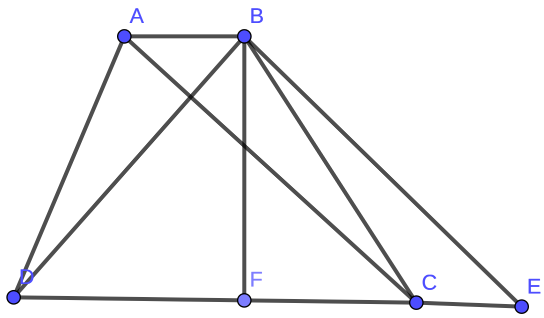
If there is $1\le n\le q$ such that $2^n\equiv 1 \pmod q$, then we are done. On the other hand, if there is no such n, then among the numbers 2^n, $1\le n\le q$, there are $q-1$ possible remainders modulo q. By the Pigeonhole Principle, there exist i and j with $1\le i<j\le q$ such that $2^i\equiv 2^j \pmod q$. Since q is odd, this implies that $2^{j-i}-1$ is a multiple of q.

3. **10 points** The height of a trapezoid $ABCD$ whose diagonals AC and BD are perpendicular is equal to 4. Find the area of the trapezoid if it is known that the length of one of its diagonals AC is equal to 5.

Solution:

We construct the parallelogram $ACEB$ as in the picture. Since the areas of the two triangles ABD and BCE are the same, the area of the trapezoid $ABCD$ is the same as that of the right triangle DBE. If BF is the height of the triangle DBE, then

$$EF^2=BE^2-BF^2=5^2-4^2=3^2.$$



On the other hand, since the two right triangles DBE and BFE are similar, we have

$$\frac{ED}{BE} = \frac{BE}{EF}$$

from which we deduce that

$$ED = \frac{BE^2}{EF} = \frac{25}{3}.$$

Therefore, the area of the triangle DBE is $\frac{1}{2}ED \cdot BF = \frac{50}{3}$. Note that the trapezoid has the same area as triangle DBE.

4. **10 points** Suppose that one removes two opposite corner squares of an 8×8 chessboard. Can the remaining 62 squares be covered with 31 dominoes (of size 2×1)? Remember to justify your answer.

Solution:

No. Opposite corner squares of a chessboard are the same color. Without loss of generality, suppose that they are both black. Removing them, we are left with 32 white and 30 black squares. As each domino covers one white square and one black square, 31 dominoes cannot cover the required squares.

5. **10 points** If n^n has 10^{100} decimal digits, how many digits does n have?

Solution:

Since n^n has 10^{100} decimal digits, $10^{10^{100}-1} \leq n^n < 10^{10^{100}}$. Let k be the number of decimal digits of n. Then $10^{k-1} \leq n < 10^k$. We will show that $k = 99$ by contradiction. If $k \geq 100$, then $n \geq 10^{99}$ and consequently

$$n^n \geq (10^{99})^{10^{99}} = 10^{99 \cdot 10^{99}} > 10^{10^{100}},$$

a contradiction. On the other hand, if $k \leq 98$, then the inequality $n < 10^{98}$ implies that

$$n^n < (10^{98})^{10^{98}} = 10^{(100-2)\cdot 10^{98}} = 10^{10^{100} - 2\cdot 10^{98}} < 10^{10^{100}-1},$$

a contradiction. Therefore, $k = 99$.

6. **10 points** Prove that there do not exist (strictly) positive integers x, y, z such that

$$x^n + y^n = z^n$$

for all integers $n \geq 3$.

Solution:

Suppose, by the way of contradiction, that there exist positive integers x, y, z such that $x^n + y^n = z^n$ for all integers $n \geq 3$. Without loss of generality, suppose that $x \geq y$. Then the fact that $z - 1 \geq x$ implies that

$$z^n = x^n + y^n \leq 2x^n \leq 2(z-1)^n$$

and consequently

$$\left(\frac{z}{z-1}\right)^n \leq 2.$$

Since $\frac{z}{z-1} > 1$, this inequality is not true when n is large.