# Department of Mathematics
# Sixth Annual
# High School Problem Solving Competition
# October 27, 2021

## Solutions

1. **10 points** Let $a$ and $b$ be integers and $\sqrt[3]{a+6x} = b$. If $a - 6 = b$, prove that $x$ is an integer.

   **Solution:** Let $a + 6x = b^3$. We substitute $a = b + 6$ and obtain

   $$b + 6 + 6x = b^3 \qquad \text{or} \qquad x = \frac{b^3 - b - 6}{6}.$$

   Since $b^3 - b = b(b-1)(b+1)$ is a product of three consecutive integers, at least one of which is even and one is a multiple of 3, we conclude $b^3 - b$ is a multiple of 6. The conclusion that $x$ is an integer follows.

2. **10 points** Find all real values of $x$ satisfying the following equation:

   $$\frac{36^x}{27^x + 64^x} + \frac{48^x}{27^x + 64^x} = \frac{1}{2}.$$

   **Solution:** We divide the top and bottom of the left side of

   $$\frac{36^x + 48^x}{27^x + 64^x} = \frac{1}{2}$$

   by $64^x$ and conclude

   $$\frac{\left(\frac{36}{64}\right)^x + \left(\frac{48}{64}\right)^x}{\left(\frac{27}{64}\right)^x + 1} = \frac{\left(\frac{9}{16}\right)^x + \left(\frac{3}{4}\right)^x}{\left(\frac{27}{64}\right)^x + 1} = \frac{\left(\frac{3}{4}\right)^{2x} + \left(\frac{3}{4}\right)^x}{\left(\frac{3}{4}\right)^{3x} + 1} = \frac{1}{2}.$$

   The substitution $u = \left(\frac{3}{4}\right)^x$ gives

   $$\frac{u^2 + u}{u^3 + 1} = \frac{u(u+1)}{(u+1)(u^2 - u + 1)} = \frac{u}{u^2 - u + 1} = \frac{1}{2}.$$

   Solving $2u = u^2 - u + 1$ yields two solutions $u = \frac{3 \pm \sqrt{5}}{2}$. Therefore,

   $$\boxed{x = \frac{\ln\left(\frac{3+\sqrt{5}}{2}\right)}{\ln\left(\frac{3}{4}\right)} \text{ or } x = \frac{\ln\left(\frac{3-\sqrt{5}}{2}\right)}{\ln\left(\frac{3}{4}\right)}}.$$

3. **10 points** Cars cross the starting line of a car race simultaneously. They arrive to the finish line one after the other with equal times elapsing among the arrival of any two cars. The average speed of the first place winner car was $v_1$, and the average speed

of the fourth place winner car was $v_4$ during the race. Express the average speed of the second place winner in terms of $v_1$ and $v_4$.

**Solution:** Let $s$ be the distance of the race. If the first place winner car completed the race in time $t_1$ and the fourth place winner in time $t_4$, then

$$t_1 = \frac{s}{v_1} \qquad \text{and} \qquad t_4 = \frac{s}{v_4}.$$

Since the times elapsing among the arrival of any two cars is the same, this elapsed time is

$$\Delta t = \frac{t_4 - t_1}{3} = \frac{\frac{s}{v_4} - \frac{s}{v_1}}{3}.$$

Therefore the time that the second place winner took to complete the race was

$$\begin{aligned} t_2 &= t_1 + \Delta t \\ &= \frac{s}{v_1} + \frac{\frac{s}{v_4} - \frac{s}{v_1}}{3} \\ &= \frac{1}{3}\left(\frac{2s}{v_1} + \frac{s}{v_4}\right) \\ &= \frac{s}{3}\left(\frac{2}{v_1} + \frac{1}{v_4}\right). \end{aligned}$$

So the average speed was

$$v_2 = \frac{s}{t_2} = \frac{3}{\frac{2}{v_1} + \frac{1}{v_4}} = \boxed{\frac{3v_1v_4}{2v_4 + v_1}}.$$

4. **10 points** Alice and Bob have unfair coins. Alice's coin lands on heads with probability 2/3 and on tails with probability 1/3. She flips her coin three times and records the sequence of outcomes: H for heads and T for tails. For example, if she gets heads on the first two flips and tails on the last flip, she will record HHT. Bob's coin lands on heads with probability 1/3 and on tails with probability 2/3. He also flips his coin three times and records the sequence of outcomes. What is the probability that Alice's sequence comes before Bob's in alphabetical order?

**Solution:** Let $S_a = x_1x_2x_3$ be the sequence of outcomes that Alice records and let $S_b = y_1y_2y_3$ be the sequence of outcomes that Bob records, where each of the $x_k$, $y_k$ is an H or a T. It will be convenient to introduce the symbol $\prec$ to denote that one string of letters precedes another in alphabetical order. (Thus, if $S_1$ and $S_2$ are two strings of letters, then $S_1 \prec S_2$ is to read "$S_1$ precedes $S_2$ alphabetically.") We will find the probability that $S_a \prec S_b$. Since all flips are independent from each other, we have

$$\begin{aligned} \text{Prob}(S_a \prec S_b) &= \text{Prob}(x_1x_2x_3 \prec y_1y_2y_3) \\ &= \text{Prob}(x_1 \prec y_1) + \text{Prob}(x_1 = y_1 \text{ and } x_2 \prec y_2) \\ &\quad + \text{Prob}(x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } x_3 \prec y_3) \\ &= \text{Prob}(x_1 \prec y_1) + \text{Prob}(x_1 = y_1) \cdot \text{Prob}(x_2 \prec y_2) \\ &\quad + \text{Prob}(x_1 = y_1) \cdot \text{Prob}(x_2 = y_2) \cdot \text{Prob}(x_3 \prec y_3). \end{aligned} \tag{*}$$

Now $x_1 \prec y_1$ is true if and only if $x_1 = \text{H}$ and $y_1 = \text{T}$. Since each of the probabilities of Alice getting heads and Bob getting tails is $2/3$, we conclude that

$$\text{Prob}(x_1 \prec y_1) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$

Similarly, $\text{Prob}(x_2 \prec y_2) = \text{Prob}(x_3 \prec y_3) = 4/9$. The relation $x_1 = y_1$ is true if and only if Alice and Bob get the same outcome, either heads or tails, on their first flip. Thus

$$\text{Prob}(x_1 = y_1) = \text{Prob}(x_1 = y_1 = \text{H}) + \text{Prob}(x_1 = y_1 = \text{T}) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$

Identical reasoning shows that $\text{Prob}(x_2 = y_2) = 4/9$ also. Substituting these probabilities into (*), we have

$$\text{Prob}(S_a \prec S_b) = \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 = \boxed{\frac{532}{729}}.$$

5. **10 points** Prove that for every integer $n > 2$,

$$(1 \cdot 2 \cdot 3 \cdot \cdots \cdot n)^2 > n^n.$$

**Solution:** The expression on the left in the inequality may be written in the form

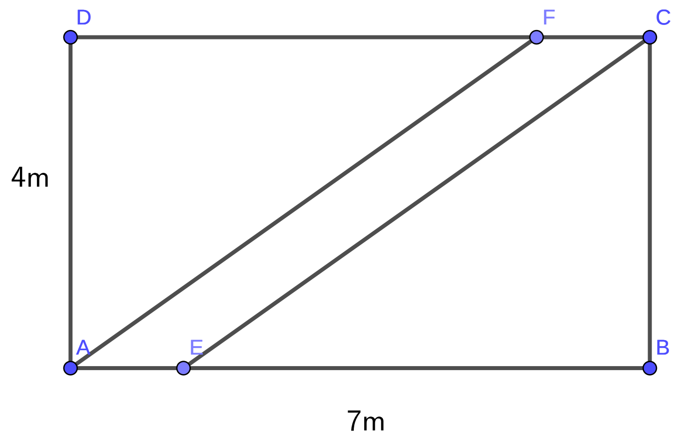
$$1 \cdot n \cdot 2 \cdot (n-1) \cdot 3 \cdot (n-2) \cdot \cdots \cdot (n-1) \cdot 2 \cdot n \cdot 1.$$

Now consider the products $1 \cdot n$, $2 \cdot (n-1)$, $3 \cdot (n-2)$, ..., $(n-2) \cdot 2$, $n \cdot 1$; these are of the form $(k+1)(n-k)$, where $k$ takes on the values $0, 1, 2, \ldots, n-1$. The first and last products are less than the others because, for $0 < k < n-1$, we have $n - k > 1$, and

$$(k+1)(n-k) = k(n-k) + (n-k) > k \cdot 1 + (n-k) = n.$$

Now the product of all these products is the expression $(1 \cdot 2 \cdot 3 \cdot \cdots \cdot n)^2$, and therefore it is greater than $n \cdot n \cdot n \cdot \cdots \cdot n = n^n$ whenever it has more than 2 factors, i.e., whenever $n > 2$. This concludes the proof.

6. **10 points** In the rectangle $ABCD$, $AB = 7$ m, $AD = 4$ m. The lines $AF$ and $EC$ are parallel, and the distance between lines $AF$ and $EC$ is 1 m. Calculate the exact area of the quadrilateral $AECF$.




**Solution:** Observe that $AECF$ is a parallelogram. Let $AE = a$, $EC = x$, both measured in meters. By the Pythagorean theorem applied to triangle $EBC$ we get that

$$x = \sqrt{4^2 + (7-a)^2}.$$

The area of the parallelogram $AECF$ can be calculated in two ways: Area $= a \cdot 4$ (taking $AE$ as base) and Area $= x \cdot 1$ (taking $EC$ as base). Therefore, $a \cdot 4 = x \cdot 1$. Substituting in the expression for $x$ and simplifying the expression we get

$$15a^2 + 14a - 65 = 0.$$

Since the negative value for $a$ does not make sense here, the only solution is $a = \frac{5}{3}$, giving the area

$$4 \text{ m} \times \frac{5}{3} \text{ m} = \boxed{\frac{20}{3} \text{ m}^2}.$$