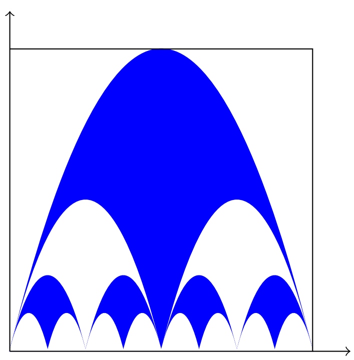
Department of Mathematics
Fifth Annual
Adnan H. Sabuwala Problem Solving Competition
November 9, 2023

Solutions

1. $\boxed{\textbf{10 points}}$ Over the interval $0 \le x \le 1$, we draw the parabola with vertex at $\left(\frac{1}{2}, 1\right)$ and passing through $(0,0)$ and $(1,0)$. We color the region between the parabola and the x-axis blue. Then we draw two "smaller" parabolas: one with vertex at $\left(\frac{1}{4}, \frac{1}{2}\right)$ and passing though $(0,0)$ and $\left(\frac{1}{2}, 0\right)$ and the other one with vertex at $\left(\frac{3}{4}, \frac{1}{2}\right)$ and passing through $\left(\frac{1}{2}, 0\right)$ and $(1,0)$. We recolor the regions between these parabolas and the x-axis white. We keep drawing parabolas in this manner. At each step, we draw twice as many parabolas as at the previous step, but they are half as "wide" and half as "tall" as the ones in the previous step, and we recolor the regions below the parabolas, alternating the color between blue and white. The result after four steps is shown in the picture below. If we continue this process for an infinite number of steps, what portion of the square $0 \le x, y \le 1$ will be colored blue?



Solution:

Since the first parabola passes through $(0,0)$ and $(1,1)$, its equation is $y = kx(x-1)$. Using the vertex $\left(\frac{1}{2}, 1\right)$, we find that $k = -4$, so the equation becomes $y = -4x(x-1) = -4x^2 + 4x$. The area of the region under this parabola is

$$\int_0^1 (-4x^2 + 4x)\, dx = \left(-\frac{4}{3}x^3 + 2x^2\right)\Big|_0^1 = -\frac{4}{3} + 2 = \frac{2}{3}.$$

The two parabolas in the second step are half as wide and half as tall, so the area under each of them is four times smaller than the area under the first parabola. Thus the blue area after two steps (that is, the area of the largest blue piece in the picture) is

$$\frac{2}{3} - 2 \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{3}.$$

The parabolas in the 3rd and 4th steps are four times smaller (in each direction) than the parabolas in the 1st and 2nd steps, respectively, so the area of each blue region obtained after 3rd and 4th steps is $\frac{1}{16}$ of $\frac{1}{3}$. However, there are four blue regions generated by the 3rd and 4th step, so their total area is $\frac{1}{4}$ of $\frac{1}{3}$. When we continue this process, after every two steps we obtain four times as many blue regions as in the previous two steps, but the area of each of them is $\frac{1}{16}$th of that of each region in the previous two steps, so the total blue area is

$$\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{4^2}\cdot\frac{1}{3}+\frac{1}{4^3}\cdot\frac{1}{3}+\cdots=\frac{\frac{1}{3}}{1-\frac{1}{4}}=\frac{4}{9}.$$

2. $\boxed{\textbf{10 points}}$

Find all real numbers a such that $a+\sqrt{2023}$ and $\frac{999}{a}+\sqrt{2023}$ are integers.

Solution:

Let $a+\sqrt{2023}=n\in\mathbb{Z}$. Then $a=n-\sqrt{2023}$. Observe that

$$\begin{aligned}\frac{999}{a}+\sqrt{2023}&=\frac{999}{n-\sqrt{2023}}+\sqrt{2023}\\&=\frac{999(n+\sqrt{2023})}{n^2-2023}+\sqrt{2023}\\&=\frac{999n+999\sqrt{2023}+n^2\sqrt{2023}-2023\sqrt{2023}}{n-\sqrt{2023}}\\&=\frac{999n+(n^2-1024)\sqrt{2023}}{n^2-2023}\\&=\frac{999n}{n^2-2023}+\frac{n^2-1024}{n^2-2023}\cdot\sqrt{2023}.\end{aligned}$$

Since $\frac{999n}{n^2-2023}$ and $\frac{n^2-1024}{n^2-2023}$ are rational, while $\sqrt{2023}$ is not (note that $2023=7\cdot17^2$ is not a perfect square), it follows that $\frac{999}{a}+\sqrt{2023}$ is rational if and only if $n^2-1024=0$. Equivalently, $n=\pm32$, so $a=\pm32-\sqrt{2023}$. Note that when $n=\pm32$, we have $\frac{999}{a}+\sqrt{2023}=\frac{999n}{n^2-2023}=\frac{999n}{-999}=-n$ is an integer.

3. $\boxed{\textbf{10 points}}$

Find an equation of the line that passes through the origin and cuts the quadrilateral with vertices at $(-20,-10)$, $(2,41)$, $(66,32)$, and $(44,-19)$ into two polygons of equal area.

Solution:

Let the vertices of the quadrilateral be A, B, C, and D, in the order listed in the problem. Observe that $\overrightarrow{AB}=(22,51)=\overrightarrow{DC}$, so the quadrilateral is a parallelogram. A line cuts a parallelogram into two polygons of equal area if it passes through its center. The center is the midpoint of the line segment joining any two opposite vertices. We find the coordinates of the midpoint of AC as $\left(\frac{-20+66}{2},\frac{-10+32}{2}\right)=(23,11)$. An equation of the line that passes through this point and the origin is $y=\frac{11}{23}x$.

4. $\boxed{\textbf{10 points}}$

   Prove that if n is a natural number, then $3n+2$ cannot have exactly seven positive factors.

   Solution:

   Observe that if $N = p_1^{\alpha_1} p_2^{\alpha_k} \dots p_k^{\alpha_k}$ is a prime factorization of a natural number N, then it has $(\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1)$ positive factors (as any positive factor of N has the form $p_1^{\beta_1} p_2^{\beta_k} \dots p_k^{\beta_k}$ where $0 \le \beta_i \le a_i$ for each $i = 1, 2, \dots, k$).

   Thus if N has exactly seven positive factors, we have $(\alpha_1+1)(\alpha_2+1)\dots(\alpha_k+1) = 7$. Since 7 is prime, $k=1$ and $\alpha_1 = 6$. In other words, $N = p^6$ for some prime p. Note that in this case $N = (p^3)^2$ is a perfect square. However, we will show that a perfect square cannot be of the form $3n+2$.

   Let $q \in \mathbb{Z}$. Then $q \equiv 0 \pmod 3$ or $q \equiv 1 \pmod 3$ or $q \equiv 2 \pmod 3$.

   If $q \equiv 0 \pmod 3$, then $q^2 \equiv 0^2 \equiv 0 \pmod 3$.

   If $q \equiv 1 \pmod 3$, then $q^2 \equiv 1^2 \equiv 1 \pmod 3$.

   If $q \equiv 2 \pmod 3$, then $q^2 \equiv 2^2 \equiv 1 \pmod 3$.

   In neither case $q^2 \equiv 2 \pmod 3$. Thus the number $3n+2$, where n is a natural number, cannot be a perfect square, and therefore, cannot have exactly seven positive factors.

5. $\boxed{\textbf{10 points}}$

   Let $f(x) : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists $0 \le x_0 \le \frac{1}{2}$ such that $f(x_0) = f\left(x_0 + \frac{1}{2}\right)$.

   Solution:

   The function

   $$g(x) = f(x) - f\left(x + \frac{1}{2}\right),\ 0 \le x \le \frac{1}{2},$$

   is continuous with

   $$g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -\left(f(0) - f\left(\frac{1}{2}\right)\right) = -g(0).$$

   By the Intermediate Value Theorem, there eixsts a point $x_0 \in \left[0, \frac{1}{2}\right]$ such that $g(x_0) = 0$. Then

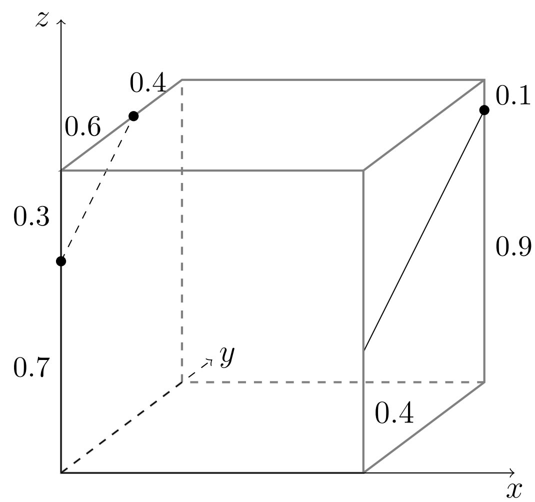
   $$f(x_0) = f\left(x_0 + \frac{1}{2}\right).$$

6. $\boxed{\textbf{10 points}}$

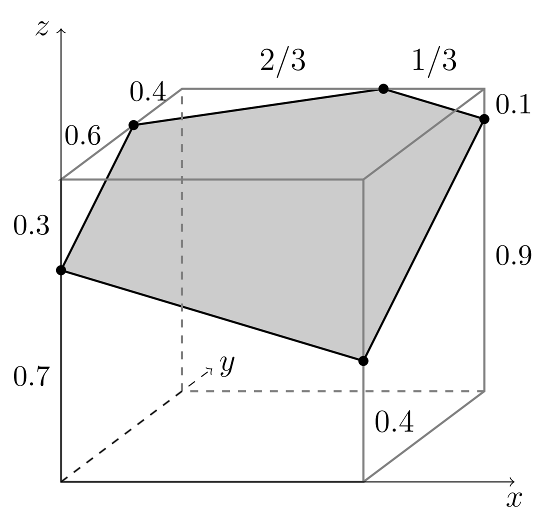
   The cube $\{(x,y,z) \mid 0 \le x,y,z \le 1\}$ has 12 edges: one joining vertices $(0,0,0)$ and $(1,0,0)$, another joining $(1,0,0)$ and $(1,0,1)$, and so on. A plane intersects a few of these edges. If $(0,0,0.7)$, $(0,0.6,1)$, and $(1,1,0.9)$ are three of such intersection points, how many more intersection points of the plane and the edges of the cube are there, and what are they?

Solution:

Let's draw the cube and the given intersection points. We see that the plane intersects the left face of the cube (lying in the $x = 0$ plane) at a line with slope $1/2$ if regarded as a line in the yz-plane. Since the right face (lying in the $x = 1$ plane) is parallel to the left face, the line of intersection with the given plane is parallel to the aforementioned line. It follows that the plane intersects the front right edge at the point $(1, 0, 0.4)$.



Using this new point and (0,0,0.7), we see that the plane intersects the front face (lying in the $y = 0$ plane) at a line with slope $-3/10$ if regarded as a line in the xz-plane. Using the same reasoning as above, the line of intersection of the back face (lying in the $y = 1$ plane) with the given plane has the same slope. Now using the point $(1, 1, 0.9)$, we obtain that the given plane intersects the back top edge at the point $\left(\frac{2}{3}, 1, 1\right)$. Observe that if we connect all these points, we obtain a pentagon all of whose sides lie on the faces of the cube, so there are no additional points of intersection of the plane with the edges of the cube.



Thus the plane intersects five edges total (front left, top left, back top, back right, and front right), and the two additional points are $(1, 0, 0.4)$ and $\left(\frac{2}{3}, 1, 1\right)$.