Department of Mathematics
Third Annual
Adnan H. Sabuwala Problem Solving Competition
November 12, 2021

Solutions

1. **10 points**

   When a natural number N is written in base 5, it has three digits. When the same number is written in base 6, it has the same three digits but in the reverse order. Find all possible values of N.

   Solution:

   Let $N = abc_5$, where a, b, and c are the digits of N in base 5. Then $N = cba_6$. Thus we have

$$\begin{aligned} abc_5 &= cba_6 \\ 25a + 5b + c &= 36c + 6b + a \\ 24a &= b + 35c \end{aligned}$$

   Since $a \leq 4$, we have $24a \leq 96$. It follows that $c \leq 2$. Since $c \neq 0$, $c = 1$ or $c = 2$. However, $c = 1$ means that $24a = b + 35$, so $a \geq 2$, but then $b \geq 13$ which is impossible. So, $c = 2$ and $24a = b + 70$. The only possible solution is $a = 3$ and $b = 2$. Thus there is only one possible value of N: $322_5 = 223_6 = 87$.

2. **10 points**

   For what values of c is there a straight line that intersects the curve

$$y = x^4 + cx^3 + 12x^2 - 5x + 2$$

   in four distinct points?

   Solution:

   Let $f(x) = x^4 + cx^3 + 12x^2 - 5x + 2$. If a line $y = ax + b$ intersects the curve $y = f(x)$ at four distinct points, then the polynomial $g(x) = f(x) - ax - b$ has four distinct roots. Since $g(x)$ has degree 4, each of its roots has multiplicity 1. Then it has two local minima and one local maximum. At the local minima, its second derivative is positive (and the graph is concave up) and at the local maximum, its second derivative is negative (and the graph is concave down). Therefore it has two inflection points, thus its second derivative has two distinct real roots. Since

$$g'(x) = 4x^3 + 3cx^2 + 24x - 5,$$

$$\begin{aligned} g''(x) &= 12x^2 + 6cx + 24 \\ &= 6(2x^2 + cx + 4), \end{aligned}$$

we see that $g''(x) = 0$ has two distinct real roots if and only if

$$\begin{aligned} c^2 - 32 &> 0 \\ c^2 &> 32 \\ |c| &> 4\sqrt{2}. \end{aligned}$$

Conversely, if $|c| > 4\sqrt{2}$, it follows that $f(x)$ has two inflection points, say, at $x = r$ and $x = s$. Let $y = ax + b$ the the line through these two inflection points. For $g(x)$ defined as above, $g''(x) = f''(x)$ is a quadratic polynomial with a positive leading coefficient, so it is negative between r and s. Therefore the local extremum of $g(x)$ between these two roots (which are inflection points of $g(x)$) is a local maximum. We will show that $g(x)$ has two distinct roots on each side of this local maximum. Suppose to the contrary that it does not. Since the value of $g(x)$ is positive at this local maximum as well as for large positive x and for large negative x, at least one of the roots ($x = r$ or $x = s$) has to have multiplicity 2. Without loss of generality, assume $x = r$ is a root of multiplicity 2. Then

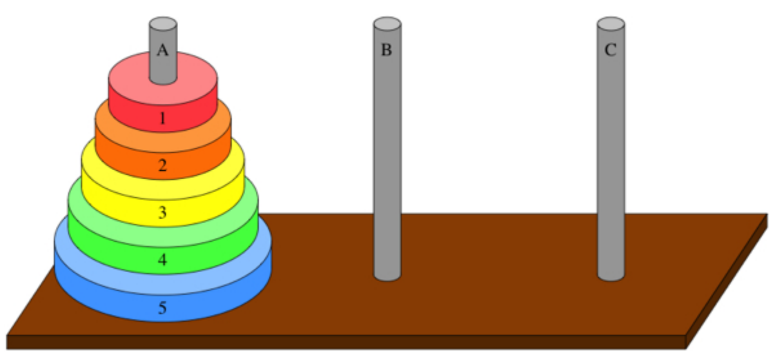
$$\begin{aligned} g(x) &= (x-r)^2(x-s)(x-t) \\ g'(x) &= 2(x-r)(x-s)(x-t) + (x-r)^2(x-t) + (x-r)^2(x-s) \\ g''(x) &= 2(x-s)(x-t) + 4(x-r)(x-t) + 4(x-r)(x-s) + 2(x-r)^2, \end{aligned}$$

where $t \neq r$. Note, however, that then $g''(r) \neq 0$, so we have a contradiction.

Thus there is a straight line that intersects the curve $y = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points if and only if $|c| > 4\sqrt{2}$.

3. **10 points**

Consider the Tower of Hanoi puzzle with three rods and five different disks.



Unlike in the actual puzzle, in this problem we will allow the disks to be placed on a rod in any order (not necessarily in the decreasing order of their radii). The order of the disks matters. For example, the arrangements shown below are considered all different.

How many arrangements are possible?

Solution:

For each such arrangements, let's write down the numbers of the disks on the first rod, starting from the bottom, then put a vertical bar to indicate that we are moving from the first rod to the second, then write the numbers of the disks on the second rod, then put another vertical bar, and finally, write the numbers of the disks on the third rod. For example, for the three arrangements shown above, we get the following:

$$4\ 2 \mid \mid 3\ 5\ 1,$$

$$2\ 4 \mid \mid 3\ 5\ 1,$$

and

$$4\ 2 \mid 3\ 5\ 1 \mid.$$

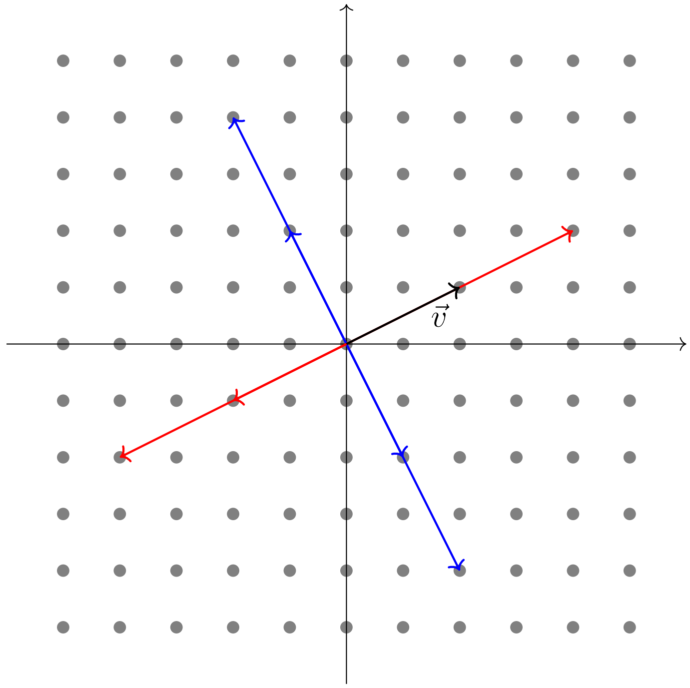
There is a one-to-one correspondence between arrangements of disks on the three rods and arrangements of numbers 1 through 5 and two vertical bars in a line. So let's count the number of arrangements of numbers 1 through 5 and two vertical bars. There are seven symbols total. From the seven positions, there are 7 ways to choose the place for 1, then there are 6 ways to choose the place for 2, 5 ways to choose the place for 3, 4 ways to choose the place for 4, and 3 ways to choose the place for 5. The two bars must be put into the remaining two places. Thus the number of arrangements is $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = \dfrac{7!}{2} = 2520$.

4. **10 points**

Each of the values of a, b, c, and d are chosen independently and randomly from the set $\{x \in \mathbb{Z} \mid -5 \leq x \leq 5\}$. Let $\vec{v} = (a, b)$, $\vec{u} = (c, d)$, and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let P_1 be the probability that $\vec{v} \cdot \vec{u} = 0$. Let P_2 be the probability that $\det(M) = 0$. What is $\dfrac{P_1}{P_2}$?

Solution:

If $\vec{v} = \vec{0}$ or $\vec{u} = \vec{0}$, then we have both $\vec{v} \cdot \vec{u} = 0$ and $\det(M) = 0$. If neither vector is $\vec{0}$, then their dot product is 0 if and only if the vectors are orthogonal, while $\det(M) = 0$ if and only if the vectors are colinear. For each $\vec{v} \neq \vec{0}$, each of the vectors colinear with it can be rotated through the angle of 90° to obtain an orthogonal vector, and all orthogonal vectors are obtained this way. In other words, there is a one-to-one correspondence between the vectors orthogonal to $\vec{v}$ and the vectors colinear with $\vec{v}$. For example, the picture below shows $\vec{v} = (2, 1)$, four vectors orthogonal to it, and four vectors colinear with it (including the vector v itself).



Then the probability that $\vec{u}$ is orthogonal to $\vec{v}$ is equal to the probability that it is colinear with $\vec{v}$. It follows that $\dfrac{P_1}{P_2} = 1$.

5. **10 points**

Let $f : [0, \infty) \to (0, \infty)$ be a continuous function and a sequence $\{x_n\}_{n=1}^{\infty}$ be recursively defined as follows:

$$x_1 = 1,$$
$$x_{n+1} = x_n \left(\frac{1}{2} + \sqrt{\frac{1}{4} + f(x_n)} \right) \text{ for all } n \in \mathbb{N}.$$

Prove that

$$\lim_{n\to\infty} x_n = \infty.$$

Solution:

As is easily verified via induction,

$$x_n > 0 \text{ for all } n \in \mathbb{N}.$$

Since

$$x_{n+1} = x_n \left(\frac{1}{2} + \sqrt{\frac{1}{4} + f(x_n)} \right) > x_n \left(\frac{1}{2} + \sqrt{\frac{1}{4} + 0} \right) = x_n \left(\frac{1}{2} + \frac{1}{2} \right) = x_n \text{ for all } n \in \mathbb{N},$$

the sequence $\{x_n\}_{n=1}^{\infty}$ is increasing.

Hence, by the Monotone Sequence Theorem, there exists

$$\lim_{n\to\infty} x_n \in [1, \infty].$$

Assume that $x = \lim_{n\to\infty} x_n$ is finite. Passing to the limit in the recursive relation as $n \to \infty$, in view of the continuity of f, we arrive at

$$x = x\left(\frac{1}{2} + \sqrt{\frac{1}{4} + f(x)}\right),$$

which implies that

$$f(x) = 0,$$

contradicting the fact that $f(x) > 0$. The obtained contradiction implies that

$$\lim_{n\to\infty} x_n = \infty.$$

6. **10 points**

How many binary strings (i.e., strings whose terms are zeros and ones) of length 10 have at least one instance of three consecutive zeros but no instances of four consecutive zeros? For example, strings 1100010101 and 0001110001 should be counted, but 1100000110 should not.

Solution:

Let's first consider the binary strings with at least one instance of three consecutive zeros. Let a_n be the number of such strings of length n. It is easy to see that $a_1 = 0$, $a_2 = 0$, $a_3 = 1$. For $n \geq 4$, consider the following cases for a string of length n.

Case 1: the string starts with 1. Then the rest of the string is any string of length $n-1$ with at least one instance of three consecutive zeros. The number of such strings is a_{n-1}.

Case 2: the string starts with 01. Then the rest of the string is any string of length $n-2$ with at least one instance of three consecutive zeros. The number of such strings is a_{n-2}.

Case 3: the string starts with 001. Then the rest of the string is any string of length $n-3$ with at least one instance of three consecutive zeros. The number of such strings is a_{n-3}.

Case 4: the strings starts with 000. Then the string already has one instance of three consecutive zeros, and the rest of the string is any string of length $n-3$. The number of such strings is 2^{n-3}.

Therefore,

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}.$$

Using this recursive formula, we obtain

$$\begin{aligned}
a_4 &= 1 + 0 + 0 + 2^1 = 3, \\
a_5 &= 3 + 1 + 0 + 2^2 = 8, \\
a_6 &= 8 + 3 + 1 + 2^3 = 20, \\
a_7 &= 20 + 8 + 3 + 2^4 = 47, \\
a_8 &= 47 + 20 + 8 + 2^5 = 107, \\
a_9 &= 107 + 47 + 20 + 2^6 = 238, \\
a_{10} &= 238 + 107 + 47 + 2^7 = 520.
\end{aligned}$$

Similarly, consider the binary strings with at least one instance of four consecutive zeros. Let b_n be the number of such strings of length n. It is easy to see that $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $b_4 = 1$. For $n \geq 5$, we use reasoning similar to the one above to obtain

$$b_n = b_{n-1} + b_{n-2} + b_{n-3} + b_{n-4} + 2^{n-4}$$

Using this recursive formula, we obtain

$$\begin{aligned}
b_5 &= 1 + 0 + 0 + 0 + 2^1 = 3, \\
b_6 &= 3 + 1 + 0 + 0 + 2^2 = 8, \\
b_7 &= 8 + 3 + 1 + 0 + 2^3 = 20, \\
b_8 &= 20 + 8 + 3 + 1 + 2^4 = 48, \\
b_9 &= 48 + 20 + 8 + 3 + 2^5 = 111, \\
b_{10} &= 111 + 48 + 20 + 8 + 2^6 = 251.
\end{aligned}$$

Therefore there are $520 - 251 = 269$ strings with at least one instance of three consecutive zeros but no instances of four consecutive zeros.