Department of Mathematics
Adnan H. Sabuwala Problem Solving Competition
November 18, 2019
Solutions

1. **10 points**

Prove that for any natural number $n > 1$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

Solution 1:

For any natural $k > 1$,

$$\frac{1}{k^2} < \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}.$$

Therefore,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} = \frac{1}{1} - \frac{1}{n} = \frac{n-1}{n}.$$

Solution 2:

The left-hand side represents the right Riemann sum for $\int_1^n \frac{1}{x^2}dx$ with $\Delta x = 1$. Since the integrand is decreasing, this Riemann sum is less than the value of the integral, which is $\int_1^n \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^n = -\frac{1}{n} + 1 = \frac{n-1}{n}$.

Note: The inequality can also be proved by Mathematical Induction.

2. **10 points**

Let a, b, c be positive integers such that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that if $\gcd(a, b, c) = 1$ (that is, there is no positive integer larger than 1 that is a factor of all three of a, b, and c), then $a + b$ is a perfect square.

Solution:

Clearly, $a > c$ and $b > c$. Let $a = c + m$ and $b = c + n$, where m, n are positive integers. Then the given equation becomes

$$c^2 = mn.$$

We will show that m and n are relatively prime. Suppose $d > 1$ is a common factor of m and n, then $c^2 = mn$ implies that d is also a factor of c, and hence of a and b. However, a, b, and c do not have a common factor larger than 1. Then $c^2 = mn$ implies that $m = k^2$ and $n = l^2$ for some integers k and l. Thus $c = kl$, and hence, $a + b = (kl + k^2) + (kl + l^2) = (k + l)^2$.

3. **10 points**

Does there exist a polynomial of degree $n > 0$ with integer coefficients

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

such that, for each non-negative integer k, $p(k)$ is a prime number?

Solution:

Assume that such a polynomial exists. Then $a_0 = p(0)$ is prime and, for each $k \in \mathbb{N}$, $p(ka_0)$ is a prime number divisible by a_0. Therefore

$$p(ka_0) = a_0 \text{ for all } k \in \mathbb{N}.$$

Then the polynomial $q(x) = p(x) - a_0$ has degree $n > 0$ and infinitely many roots. This contradicts the Fundamental Theorem of Algebra. Thus such a polynomial $p(x)$ does not exist.

4. **10 points**

Let $S = \{1, 2, 3, 4, 5\}$. Suppose that a relation on S is chosen at random. What is the probability that it is an equivalence relation?

Solution:

A relation on S is any subset of $S \times S$. Since $S \times S$ has 25 elements, it has 2^{25} subsets (as each element has two choices: it either is in a subset or is not in it). Thus there are 2^{25} relations on S total.

An equivalence relation is uniquely determined by the set of its equivalence classes, which is a partition of the set S. We will count the number of partitions of S.

There is only one partition consisting of one subset, namely, $\{S\}$.

If a partition consists of two subsets, then these subsets contain either two and three elements, or one and four. There are $\binom{5}{2} = 10$ ways to choose two elements out of five, and there are five ways to choose one element. Thus the number of such partitions is $10 + 5 = 15$.

If a partition consists of three subsets, then these subsets contain either three, one, and one elements, or two, two, and one. There are $\binom{5}{3} = 10$ ways to select three elements out of five, and there are $\frac{\binom{5}{2}\cdot\binom{3}{2}}{2} = 15$ partitions of the second type (as there are $\binom{5}{2}$ ways to select the first subset with two elements, and then $\binom{3}{2}$ ways to select the second subset with two elements, but we counted each pair twice). Thus the number of such partitions is $10 + 15 = 25$.

If a partition consists of four subsets, then they contain two, one, one, and one elements, and there are $\binom{5}{2} = 10$ ways to select the subset with two elements.

Finally, there is one partition consisting of five subsets.

The total number of partitions is then $1 + 15 + 25 + 10 + 1 = 52$.

Therefore the probability that a randomly selected relation is an equivalence relation is $\frac{52}{2^{25}} = \frac{13}{2^{23}}$.

5. **10 points**

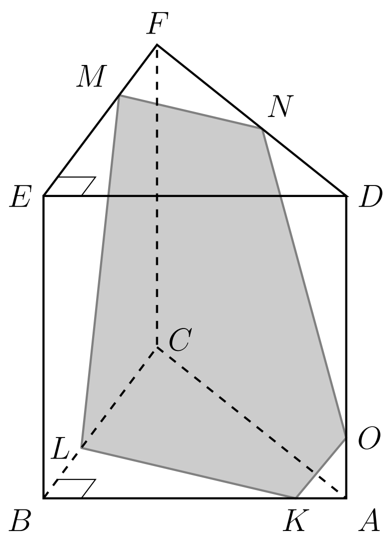
Suppose S is a set with 2019 elements. How many pairs (A, B) of nonempty disjoint subsets of S are there?

Solution:

Each element of S has three choices: it can be in A, B, or $S - (A \cup B)$. Since there are 2019 elements, we have 3^{2019} choices total, so there are 3^{2019} pairs (A, B) of disjoint subsets of S. This includes the pairs where A or B is empty. If A is empty, each element has two choices: it can be in B or $S - B$. So there are 2^{2019} pairs with $A = \emptyset$. Similarly, there are 2^{2019} pairs with $B = \emptyset$. However, the pair $(\emptyset, \emptyset)$ was counted twice, so the number of pairs with A or B empty is $2 \cdot 2^{2019} - 1 = 2^{2020} - 1$. Thus the number of nonempty disjoint pairs (A, B) is $3^{2019} - 2^{2020} + 1$.

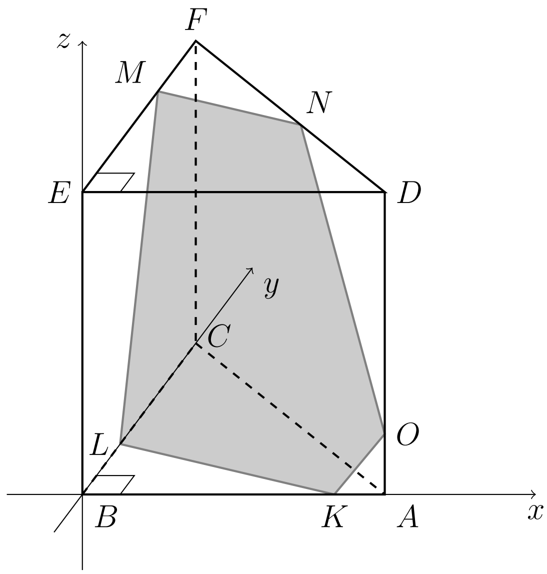
6. **10 points**

The base ABC of a right triangular prism $ABCDEF$ is a right isosceles triangle with $\angle B = 90°$, and the height of the prism is $AD = AB$. A plane intersects this prism in a pentagon $KLMNO$ so that K, L, M, N, and O lie on AB, BC, EF, DF, and AD, respectively. Also, $BL : LC = 1 : 2$, $EM : MF = 2 : 1$, and $AO : OD = 1 : 4$. Find $DN : NF$.



Solution 1:

Introduce the coordinate system so that B is at the origin, and the other vertices of the prism have coordinates as follows: $A(1, 0, 0)$, $C(0, 1, 0)$, $D(1, 0, 1)$, $E(0, 0, 1)$, and $F(0, 1, 1)$. Then the coordinates of L, M, and O follow from the given ratios, namely, $L(0, \frac{1}{3}, 0)$, $M(0, \frac{2}{3}, 1)$, and $O(1, 0, \frac{1}{5})$.



Let us write an equation of the plane through the points L, M, and O. Since $\overrightarrow{LM} = \left(0, \frac{1}{3}, 1\right)$ and $\overrightarrow{LO} = \left(1, -\frac{1}{3}, \frac{1}{5}\right)$, a normal vector to the plane is

$$\mathbf{n} = \overrightarrow{LM} \times \overrightarrow{LO} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{3} & 1 \\ 1 & -\frac{1}{3} & \frac{1}{5} \end{vmatrix} = \frac{2}{5}\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k} = \left(\frac{2}{5}, 1, -\frac{1}{3}\right).$$

Hence we can write an equation of the plane as

$$\frac{2}{5}x + \left(y - \frac{1}{3}\right) - \frac{1}{3}z = 0.$$

Point N, since it lies on line DF, has coordinates $(x, 1-x, 1)$. Since it lies in the plane, its coordinates must satisfy the equation of the plane, thus

$$\frac{2}{5}x + \left(1 - x - \frac{1}{3}\right) - \frac{1}{3} = 0.$$

Solving this equation gives $x = \frac{5}{9}$. It follows that $DN : NF = 4 : 5$.

Solution 2:

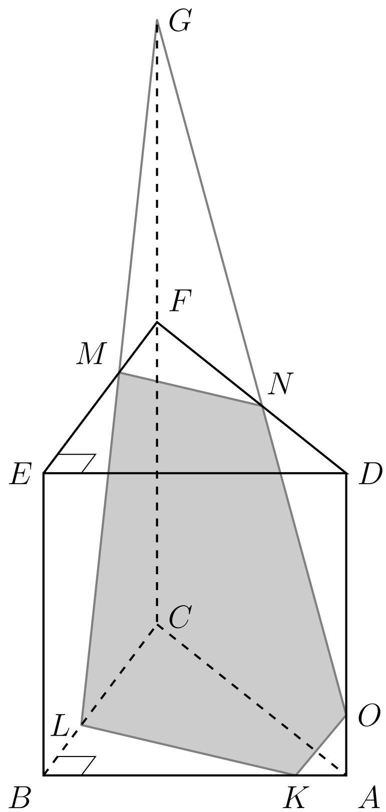
Introduce the coordinate system as in Solution 1, and let point N have coordinates $(x, 1-x, 1)$. Then vectors $\overrightarrow{LM} = \left(0, \frac{1}{3}, 1\right)$, $\overrightarrow{LO} = \left(1, -\frac{1}{3}, \frac{1}{5}\right)$, and $\overrightarrow{LN} = \left(x, \frac{2}{3} - x, 1\right)$ are linearly dependent. Let us consider scalar multiples of these vectors to ease the calculation: $(0, 1, 3)$, $(15, -5, 3)$, and $(3x, 2-3x, 3)$. The determinant of the matrix with these rows must be 0:

$$\begin{vmatrix} 0 & 1 & 3 \\ 15 & -5 & 3 \\ 3x & 2-3x & 3 \end{vmatrix} = -81x + 45 = 0,$$

so $x = \frac{5}{9}$. It follows that $DN : NF = 4 : 5$.

Solution 3:

Extend line segments LM and CF until they intersect, and let G be their intersection point. This point lies in the plane, and, in fact, is the only intersection point of the plane with line containing CF (since this line does not belong to the plane). Thus if ON is extended until it meets this line, it must meet the line at G as well.



Triangles GMF and GLC are similar with $MF : LC = 1 : 2$, so $GF : GC = 1 : 2$ as well, therefore $GF = CF = AD$. Triangles ODN and GFN are also similar with $OD : GF = \frac{4}{5} : 1$, so $DN : NF = \frac{4}{5} : 1 = 4 : 5$.