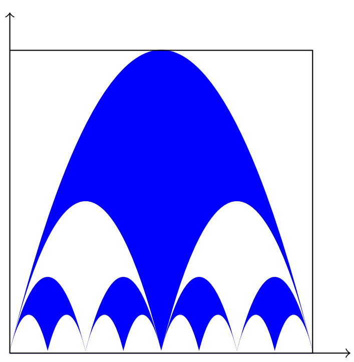
Department of Mathematics
Fifth Annual
Adnan H. Sabuwala Problem Solving Competition
November 9, 2023

1. **10 points** Over the interval $0 \leq x \leq 1$, we draw the parabola with vertex at $\left(\frac{1}{2}, 1\right)$ and passing through $(0,0)$ and $(1,0)$. We color the region between the parabola and the x-axis blue. Then we draw two "smaller" parabolas: one with vertex at $\left(\frac{1}{4}, \frac{1}{2}\right)$ and passing though $(0,0)$ and $\left(\frac{1}{2}, 0\right)$ and the other one with vertex at $\left(\frac{3}{4}, \frac{1}{2}\right)$ and passing through $\left(\frac{1}{2}, 0\right)$ and $(1,0)$. We recolor the regions between these parabolas and the x-axis white. We keep drawing parabolas in this manner. At each step, we draw twice as many parabolas as at the previous step, but they are half as "wide" and half as "tall" as the ones in the previous step, and we recolor the regions below the parabolas, alternating the color between blue and white. The result after four steps is shown in the picture below. If we continue this process for an infinite number of steps, what portion of the square $0 \leq x, y \leq 1$ will be colored blue?



2. **10 points**

   Find all real numbers a such that $a + \sqrt{2023}$ and $\dfrac{999}{a} + \sqrt{2023}$ are integers.

3. **10 points**

   Find an equation of the line that passes through the origin and cuts the quadrilateral with vertices at $(-20, -10)$, $(2, 41)$, $(66, 32)$, and $(44, -19)$ into two polygons of equal area.

4. **10 points**

   Prove that if n is a natural number, then $3n + 2$ cannot have exactly seven positive factors.

5. **10 points**

   Let $f(x) : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists $0 \leq x_0 \leq \frac{1}{2}$ such that $f(x_0) = f\left(x_0 + \frac{1}{2}\right)$.

6. **10 points**

   The cube $\{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ has 12 edges: one joining vertices $(0, 0, 0)$ and $(1, 0, 0)$, another joining $(1, 0, 0)$ and $(1, 0, 1)$, and so on. A plane intersects a few of these edges. If $(0, 0, 0.7)$, $(0, 0.6, 1)$, and $(1, 1, 0.9)$ are three of such intersection points, how many more intersection points of the plane and the edges of the cube are there, and what are they?