

Department of Mathematics
Adnan H. Sabuwala Problem Solving Competition
November 18, 2019

1. **10 points**

Prove that for any natural number $n > 1$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$

2. **10 points**

Let a, b, c be positive integers such that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that if $\gcd(a, b, c) = 1$ (that is, there is no positive integer larger than 1 that is a factor of all three of a, b , and c), then $a + b$ is a perfect square.

3. **10 points**

Does there exist a polynomial of degree $n > 0$ with integer coefficients

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

such that, for each non-negative integer k , $p(k)$ is a prime number?

4. **10 points**

Let $S = \{1, 2, 3, 4, 5\}$. Suppose that a relation on S is chosen at random. What is the probability that it is an equivalence relation?

5. **10 points**

Suppose S is a set with 2019 elements. How many pairs (A, B) of nonempty disjoint subsets of S are there?

6. **10 points**

The base ABC of a right triangular prism $ABCDEF$ is a right isosceles triangle with $\angle B = 90^\circ$, and the height of the prism is $AD = AB$. A plane intersects this prism in a pentagon $KLMNO$ so that $K, L, M, N,$ and O lie on $AB, BC, EF, DF,$ and AD , respectively. Also, $BL : LC = 1 : 2$, $EM : MF = 2 : 1$, and $AO : OD = 1 : 4$. Find $DN : NF$.

