Department of Mathematics
Third Annual Problem Solving Contest
November 16, 2017

Name: ____

Email: ____

1. **10 points**

   Real numbers a and b are chosen randomly and independently in the interval $[-1, 1]$. Find the probability that the line $y = ax + b$ and the parabola $y = x^2$ intersect.

2. **10 points**

   Let a, b, c be real numbers with $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, and $a + b + c = 2$. Prove that

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$

3. **10 points**

   A unit cube is projected onto a plane. Prove that the sum of the squares of the lengths of the projections of all its edges is equal to 8.

4. **10 points**

   Let $A = \{1, 2, 3, \ldots, 100\}$ and B be a subset of A having 48 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

5. **10 points**

   Determine all non-negative integral pairs (x, y) for which

$$(xy - 7)^2 = x^2 + y^2.$$

6. **10 points**

Prove that the sequence

$$x_n = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\dots \sqrt{2 + \sqrt{2}}}}}} \quad (n \text{ roots}),\ n \in \mathbb{N},$$

is convergent and find $\lim_{n \to \infty} x_n$.