2022
Leap Frog Relay Grades 9-10

Solutions - Part I

1. How many (distinct) real number solutions are there to the equation

$$x^{2022} - 4x^{2020} = x^{2020} - 4x^{2018} \text{ ?}$$

(a) 2

(b) 3

(c) 4

(d) 5

(e) None of these

Solution. (d) Factoring both sides of the equation gives:

$$x^{2020}(x^2 - 4) = x^{2018}(x^2 - 4).$$

The choice $x = 0$ is a solution. If $x \neq 0$, then we may divide both sides by x^{2018} and get

$$x^2(x^2 - 4) = x^2 - 4.$$

If $x = \pm 2$, we have two more solutions. If $x \neq \pm 2$, we may divide both sides by $(x^2 - 4)$ and obtain

$$x^2 = 1,$$

which has two solutions $x = \pm 1$. Therefore we have 5 total solutions: $0, \pm 1, \pm 2$.

2. A palindromic number is a number that is exactly the same if the order is reversed; for example, 1225221 or 1331. How many four-digit palindromic numbers are there for which the sum of its digits is divisible by four?

(a) 20

(b) 45

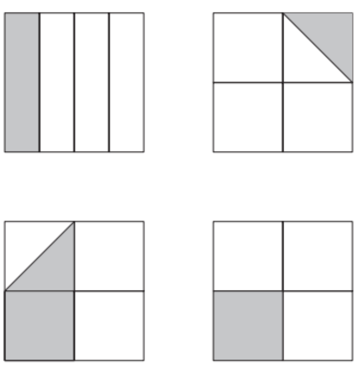
(c) 50

(d) 100

(e) None of these.

Solution. **(b)** For a four-digit palindromic number $abba$, there are nine possible choices for a: $1, 2, \ldots, 9$; zero will make the number a three digit number. The sum of digits is $2a+2b = 2(a+b)$, and this sum is divisible by 4 if $a + b$ is divisible by 2. If a is even, there are four possibilities for a, namely $2, 4, 6, 8$. For $a + b$ to be divisible by two, b must be even as well, so there are five possible choices for b: $0, 2, 4, 6, 8$. This totals to 20 possible choices when a is even. If a odd, there are five possibilities $1, 3, 5, 7, 9$ for the value of a. To be divisible by 2, b must also be odd, with the same possible choices. This totals to 25 possible choices when a is odd. Therefore, there are $20 + 25 = 45$ possible such palindromic numbers.

3. Each of the following four squares are the same size. Each square is subdivided into congruent rectangles or congruent triangles. What percentage of the total area is shaded?



(a) 12.5

(b) 20

(c) 25

(d) $33\frac{1}{3}$

(e) 37.5

Solution. **(c)** The upper left and lower right squares have $1/4$ of their area shaded. If we remove the shaded triangle from the lower left square and add it to the upper right square, those both now have $1/4$ of their area shaded. Therefore, $1/4 = 25$ % have their area shaded.

4. If $3^p + 3^4 = 90$, $2^r + 44 = 76$, and $5^3 + 6^s = 1421$, what is the product of p, r, and s?

(a) 27

(b) 40

(c) 50

(d) 70

(e) 90

Solution. (b) For p, $3^p + 81 = 90$, so $3^p = 9$; thus $p = 2$. For $2^r + 44 = 76$ we have $2^r = 32$, so $r = 5$. For $125 + 6^s = 1421$, we get $6^s = 1296$, so $s = 4$. Then $prs = 2 * 5 * 4 = 40$.

5. Jamal drives from his apartment to the airport to catch a flight. He drives 35 miles in the first hour, but realizes he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

(a) 149

(b) 175

(c) 180

(d) 210

(e) 245

Solution. (d) Jamal is going at 35 mph the first hour. Let d be the remaining distance after one hour of driving and let t be the remaining time. Then $d = 35(t + 1)$ and $d = 50(t - .5)$. Then solving this gives $t = 4$ and $d = 175$. Adding the initial 35 miles gives a total distance of 210 miles.

6. Sarah buys 4 muffins and 3 bananas. Chris spends twice as much on 2 muffins and 16 bananas. A muffin is how many times more expensive than a banana?

(a) $\frac{3}{2}$

(b) $\frac{5}{3}$

(c) $\frac{7}{4}$

(d) 2

(e) $\frac{13}{4}$

Solution. (b) Let a muffin cost m dollars and a banana cost b dollars. Then $2(4m+3b) = 2m+16b$. Therefore $m = \frac{5}{3}b$.

7. A parabola $y = ax^2 + bx + c$ has its vertex at $(6, 15)$ and contains the point $(0, -3)$. What is the product abc?

(a) $-\dfrac{4}{9}$

(b) -1

(c) 4

(d) 9

(e) None of these

Solution. (d) Substituting $(0, -3)$ into the equation tells us that $c = -3$. The x-coordinate of the vertex is $-b/2a = 6$ and thus $b = -12a$. Substituting this and the vertex $(6, 15)$ into the equation for the parabola, we have

$$\begin{aligned} 15 &= a \cdot 6^2 - 12a \cdot 6 - 3 \\ 18 &= -36a \\ a &= -\frac{1}{2} \\ b &= 6 \quad (\text{since } b = -12a) \end{aligned}$$

Therefore, the product abc is: $abc = (-\frac{1}{2}) \cdot (6) \cdot (-3) = 9$.

8. What is the value of $4 * (-1 + 2 - 3 + 4 - 5 + 6 - \cdots - 999 + 1000)$?

(a) -10

(b) 0

(c) 1

(d) 500

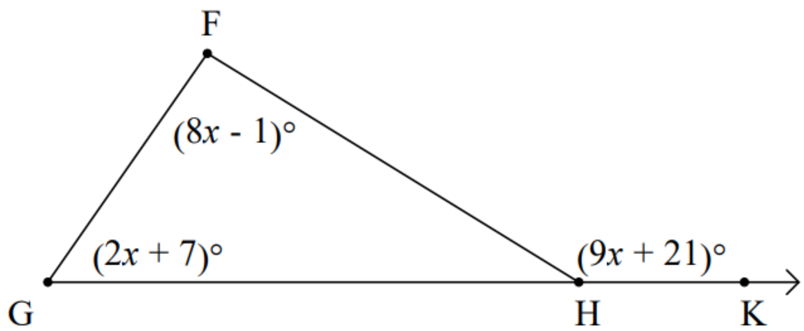
(e) 2000

Solution. (e) In the parentheses, pairing off terms results in

$$(-1 + 2) + (-3 + 4) + (-5 + 6) + \cdots + (-999 + 1000).$$

The number in each set of parentheses is equal to 1, and there are 500 of them. Multiplying by 4, we get 2000.

9. What is the measure of the angle $\angle FGH$?



(a) 15°

(b) 22°

(c) 37°

(d) 57°

(e) 119°

Solution. **(c)** The angle $\angle FHG$ is supplementary to $(9x+21)^\circ$, so it has measure $180^\circ-(9x+21)^\circ$. The interior angles of a triangle must sum to 180°, so $(2x+7)^\circ+(8x-1)^\circ+180^\circ-(9x+21)^\circ = 180^\circ$. Solving for x gives 15°. So $m\angle FGH = (2 * 15 + 7)^\circ = 37^\circ$.

10. If $9^{5x+3} = 27^{2x+1}$, then $81^x = ?$

(a) $-\dfrac{3}{4}$

(b) $\dfrac{1}{27}$

(c) $\dfrac{1}{9}$

(d) $\dfrac{1}{3}$

(e) None of These

Solution. **(b)** The equation can be rewritten in base 3: $3^{10x+6} = 3^{6x+3}$. Exponential functions are one-to-one, so $10x + 6 = 6x + 3$. Solving for x, we have

$$\begin{aligned} 10x + 6 &= 6x + 3 \\ 4x &= -3 \end{aligned}$$

Then $81^x = 3^{4x} = 3^{-3} = \dfrac{1}{3^3} = \dfrac{1}{27}$.

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Solutions - Part II

11. Suppose
$$\frac{x^3 + x^2 + cx + d}{x + 2} = ax^2 + bx + 4$$
for all x except $x = -2$. What is $3^2a + 3b + 4$?

(a) $\frac{38}{5}$

(b) 9

(c) 10

(d) $\frac{54}{5}$

(e) None of the above

Solution. (c) The equation becomes
$$\begin{aligned} x^3 + x^2 + cx + d &= (ax^2 + bx + 4)(x + 2) \\ x^3 + x^2 + cx + d &= ax^3 + (2a + b)x^2 + (4 + 2b)x + 8. \end{aligned}$$

Matching coefficients, we have
$$\begin{aligned} x^3 :& \quad 1 = a \\ x^2 :& \quad 1 = 2a + b \\ x^1 :& \quad c = 4 + 2b \\ x^0 :& \quad d = 8 \end{aligned}$$

so $a = 1,\ b = -1,\ c = 2,\ d = 8$. Then $3^2a + 3b + 4 = 9 - 3 + 4 = 10$.

12. The lines $y = 2x + b$ and $y = x + 2022$ meet at a point on the line $y = 4x + 18$. Determine b.

(a) 1250

(b) 1251

(c) 1351

(d) 1350

(e) None of the above

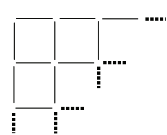
Solution. (e) We note that the three lines must meet at a common point, which is the point where the first two lines intersect. Set the second two equations equal to each other. We have:

$$\begin{aligned} x + 2022 &= 4x + 18 \\ 2004 &= 3x \\ x &= 668. \end{aligned}$$

Set the first two equations equal to each other to solve for b:

$$\begin{aligned} 2x + b &= x + 2022 \\ b &= -x + 2022 \\ b &= -668 + 2022 \quad \text{(by plugging 668 into } x\text{)} \\ b &= 1354. \end{aligned}$$

13. A grid is formed with toothpicks that is 60 toothpicks across and 32 toothpicks high. How many toothpicks are needed to form this grid?



(a) 1920

(b) 2952

(c) 2022

(d) 1980

(e) 3932

Solution. (e) The grid will need 61 columns of 32 toothpicks and 33 rows of 60 toothpicks. The total is $(61 \times 32) + (33 \times 60) = 3932$.

14. If cars hold 5 passengers and charge \$29 for a trip to the airport and vans hold 7 passengers and charge \$41 for a trip to the airport, find the minimum cost to transport 49 people to the airport.

(a) \$280
(b) \$285
(c) \$287
(d) \$290
(e) None of the above

Solution. **(b)** The cost per seat for the car is $\$29/5 = \$5.80$ and for the van is $\$41/7 \approx \$5.86$, so the cars are cheaper. If they took 10 cars, it would cost $\$29 * 10 = \$290$. If they took 7 vans, it would cost $\$41 * 7 = \$287$. So not filling all the seats is not cost effective. They could take 7 cars and 2 vans without wasting a seat, costing $\$29 * 7 + \$41 * 2 = \$203 + \$82 = \$285$, which is the minimum cost.

15. Define the operation \$ by $a\$b = a + b - ab$. So for example $7\$10 = 7 + 10 - 70 = -53$. What is $(2\$3)\$4$?

(a) 4
(b) 5
(c) 6
(d) 7
(e) None of the above

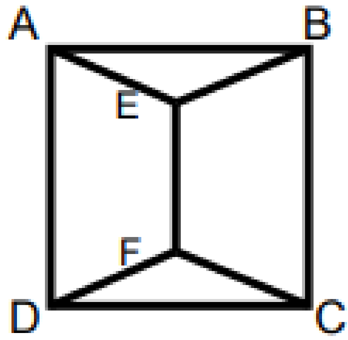
Solution. **(d)** By order of operations, we must do the operation in the parentheses first. We have $2\$3 = 2 + 3 - 2 * 3 = -1$. Then $(2\$3)\$4 = (-1)\$4 = -1 + 4 - (-1)(4) = 7$.

16. A fair coin is tossed 3 times. What is the probability of at least two consecutive heads?

(a) $\frac{1}{8}$
(b) $\frac{1}{4}$
(c) $\frac{3}{8}$
(d) $\frac{1}{2}$
(e) $\frac{3}{4}$

Solution. **(c)** The list of possibilities is HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. Of the eight possibilities, only three have two consecutive heads, so the answer is $3/8$.

17. Suppose the square ABCD has side length 6. Suppose all interior segments AE, BE, DF, CF, and EF have the same length. Find the length of EF.



(a) $\frac{6+\sqrt{15}}{2}$

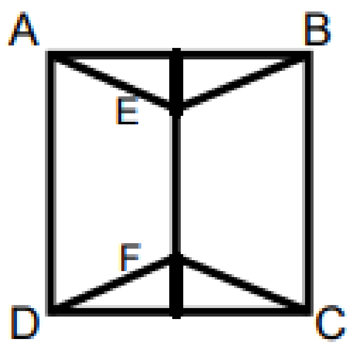
(b) $2\sqrt{7}-2$

(c) $\frac{79}{24}$

(d) $2(\sqrt{2}+\sqrt{3})$

(e) None of the above

Solution. (b) Let EF have length x. Then all the interior line segments have length x. From F, drop a height with length h as below.



Vertically, $2h + x$ is equal to the side length of the square, which is 6. Solving for h we have $h = 3 - x/2$. We also have that the two right triangles on the bottom have legs of length 3 and h and hypotenuse of length x. By the Pythagorean theorem, $9 + h^2 = x^2$. Substituting the expression for h, we have $18 - 3x + x^2/4 = x^2$ which simplifies to $\frac{3}{4}x^2 + 3x - 18 = 0$ and then to $x^2 + 4x - 24 = 0$. Using the quadratic formula, we have

$$x = \frac{-4 \pm \sqrt{112}}{2} = -2 \pm 2\sqrt{7}.$$

Since the length is positive, the answer is $-2 + 2\sqrt{7}$.

18. At the start of class all thirty students are awake. During the class, students fall asleep at a rate of one student every thirty seconds. Ten minutes into the class, a sleeping student wakes up. The sleeping students continue to wake up at a rate of one student every minute. Any student who woke up can fall back asleep. How many minutes to the first time in class that all students will be asleep?

(a) 19.5

(b) 20.5

(c) 22.5

(d) 25.5

(e) 30.5

Solution. (b) At ten minutes, 19 students are asleep, 20 fell asleep and 1 woke up. At 10.5 minutes, 20 students are asleep. At 11 minutes, 20 students are asleep since one fell asleep and one awoke from 10.5 minutes. So every minutes, two students fall asleep and one awake, increasing the total number asleep by one per minute. So at 20 minutes, there are 29 asleep; thus the 30^{th} student falls asleep at 20.5 minutes.

19. What is the ones digit for the product $(5+1)(5^3+3)(5^6+6)(5^{12}+12)$?

(a) 0

(b) 2

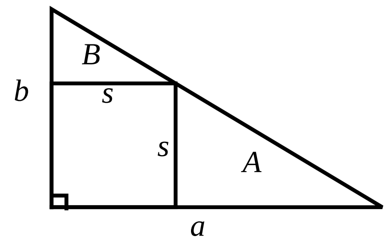
(c) 4

(d) 6

(e) 8

Solution. (d) When multiplying, the ones digits of the product only depends on the ones digit of the terms, so we may ignore numbers from the tens digit and up. For the ones digit, 5 to any power has a 5 in the ones digit. So for the ones digit, $(5+1)(5^3+3)(5^6+6)(5^{12}+12)$ is the same as $(5+1)(5+3)(5+6)(5+2)$ is the same as $(6)(8)(1)(7) = (48) * 7$ which is the same as $8 * 7 = 56$ which has ones digit 6.

20. In the figure below, the large right triangle has respective leg lengths a and b, as pictured. The s by s square is inscribed in the triangle. The respective areas of the two smaller right triangles are A and B as indicated. Determine the ratio of the areas A/B as a function of a and b.



(a) $A/B = a^2/b^2$

(b) $A/B = a/b$

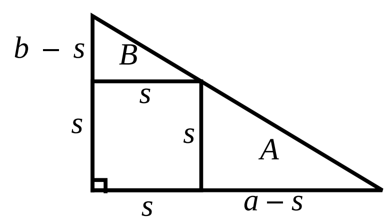
(c) $A/B = (ab)/(a+b)$

(d) $A/B = \sqrt{a^2+b^2}/(a+b)$

(e) None of the above

Solution. **(a)** If we compare the similar triangle pair that is the small top triangle with the large triangle, we get equal ratios:

$$\frac{b-s}{s} = \frac{b}{a} \Longrightarrow s = \frac{ab}{a+b}.$$



So,

$$\begin{aligned}\frac{A}{B} &= \frac{\frac{1}{2}(a-s)s}{\frac{1}{2}(b-s)s} \\ &= \frac{a-s}{b-s} \\ &= \frac{a-\frac{ab}{a+b}}{b-\frac{ab}{a+b}} \\ &= \frac{a^2}{b^2}.\end{aligned}$$