2022 Leap Frog Relay Grades 11-12 Part I — Problems 1-10

No calculators allowed

Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. How many ordered pairs of integers (x, y) are there so that

$$0 < |xy| < 36?$$

(a) 6

(b) 5

(c) 524

(d) 1225

(e) 131

2. If a, b, c, d, and e are constants such that <u>for every $x > 0$</u> we get

$$\frac{5x^4 - 4x^3 + 3x^2 - 2x + 1}{(x+2)^4} = a + \frac{b}{x+2} + \frac{c}{(x+2)^2} + \frac{d}{(x+2)^3} + \frac{e}{(x+2)^4}$$

then what is the value of $a + b + c + d + e$?

(a) 15

(b) 5

(c) 3

(d) 1

(e) It cannot be determined.

3. The Fibonacci numbers are defined by taking $F_0 = 0$ and $F_1 = 1$ and then, for $n \geq 2$, recursively by the equation

$$F_n = F_{n-1} + F_{n-2}.$$

For every $n \geq 0$, define

$$T_n = F_{3n}.$$

Note that this new sequence of numbers considers only every third Fibonacci number. Assume that there are constants a and b so that for every integer $n \geq 2$ we get

$$T_n = aT_{n-1} + bT_{n-2}.$$

Find these numbers a and b.

(a) $a = 1$ and $b = 1$

(b) $a = 1$ and $b = 2$

(c) $a = 2$ and $b = 1$

(d) $a = 1$ and $b = 4$

(e) $a = 4$ and $b = 1$

4. The admission tickets for a local band concert cost \$25 for admission with a VIP pass or \$12 for a regular admission ticket. Last Saturday, the band collected \$1950 in admission fees from at least one VIP pass and at least one regular admission. Of all the possible ratios of VIPs to non-VIPs at the concert last Saturday, which one is closest to 1? Express your answer as a fraction in reduced form.

(a) $\frac{12}{25}$

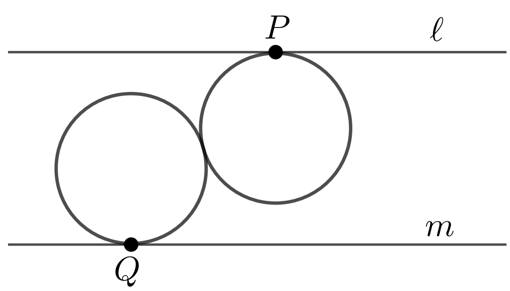
(b) $\frac{53}{52}$

(c) $\frac{27}{25}$

(d) $\frac{25}{12}$

(e) It cannot be determined.

5. Consider two circles tangent to two parallel lines, ℓ and m, at points P and Q respectively. Assume that the circles are tangent to each other, as shown in the picture below. If the two circles have the same radius, r, the distance between the lines is 12 in., and the distance between P and Q is 13 in., what is the value of r?



(a) $\frac{5}{2}$ in.

(b) 6 in.

(c) $\frac{13}{2}$ in.

(d) $\frac{144}{65}$ in.

(e) $\frac{169}{48}$ in.

6. Consider a fair coin and a fair 6-sided die. The die begins with the number 1 face up. A step starts with a toss of the coin: if the coin comes out heads, we roll the die; otherwise we do nothing. After 5 such steps, what is the probability that the number 1 is face up on the die? Express your answer as a fraction in reduced form.

(a) $\frac{11}{192}$

(b) $\frac{37}{192}$

(c) $\frac{1}{7776}$

(d) $\frac{9331}{248832}$

(e) $\frac{16807}{248832}$

7. It is given that $n = (\sqrt{3}+5)^{103} - (\sqrt{3}-5)^{103}$ is an integer. What is its remainder when it is divided by 9?

(a) 0

(b) 1

(c) 3

(d) 5

(e) 7

8. How many ordered triples of integers (x, y, z) are there such that

$$x^2 + y^2 + z^2 = 34?$$

(a) 2

(b) 9

(c) 24

(d) 48

(e) 72

9. Consider the sequence of six real numbers $60, 10, 100, 150, 30$, and x. The average of the terms of this sequence is equal to the median of the sequence. What is the sum of all the possible values of x?
Note: The median of a sequence of six real numbers is the average of the two middle numbers after all the numbers have been arranged in increasing order.

(a) -80

(b) 125

(c) 130

(d) 135

(e) 215

10. $\triangle ABC$ lies on the coordinate plane. The midpoint of $\overline{AB}$ has coordinates $(-16, -63)$, the midpoint of $\overline{AC}$ has coordinates $(13, 50)$, and the midpoint of $\overline{BC}$ has coordinates $(6, -85)$. What are the coordinates of point A?

(a) $(-18, 144)$

(b) $(-9, 72)$

(c) $(-3, -13)$

(d) $(3, -98)$

(e) $(6, -85)$

2022 Leap Frog Relay Grades 11-12 Part II — Problems 11-20

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11. Let x and y be positive integers such that $x^4 - y^4 = 3439$. What is the value of xy?
Hint: $3439 = 19 \cdot 181$.

(a) 1

(b) 19

(c) 90

(d) 181

(e) $\sqrt[4]{3439}$

12. Let $f(x)$ be a real-valued function defined for all real numbers $x \neq 0, 1$ such that

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x.$$

Find $f(2)$.

(a) $\frac{3}{4}$

(b) 1

(c) $\frac{3}{2}$

(d) $\frac{5}{2}$

(e) 3

13. Define a set of positive integers to be *balanced* if the set is not empty and the number of even integers in the set is equal to the number of odd integers in the set. How many strict subsets of the set of the first 10 positive integers are balanced?

(a) 4

(b) 10

(c) 58

(d) 125

(e) 250

14. Find real numbers a and b so that $a < k < b$ if and only if $x^3 + \frac{1}{x^3} = k$ does not have a real solution.

(a) $a = -8$ and $b = 8$

(b) $a = -\sqrt[3]{2}$ and $b = \sqrt[3]{2}$

(c) $a = -2$ and $b = 2$

(d) $a = 0$ and $b = -\infty$

(e) The equation always has real solutions

15. An isosceles right triangle is inscribed in a circle of radius 5 in., thereby separating the circle into four regions. Compute the sum of the areas of the two smallest regions.

(a) $\frac{25}{2}\text{in.}^2$

(b) $25\left[\frac{\pi}{2} - 1\right]\text{ in.}^2$

(c) $25\left[\frac{\pi}{2} + 1\right]\text{ in.}^2$

(d) $\frac{25\pi}{4} - \frac{25}{2}\text{in.}^2$

(e) $\frac{25\pi}{4}\text{in.}^2$

16. For a positive integer a, let $f(a)$ be the average of all positive integers b such that $x^2 + ax + b = 0$ has integer solutions. Compute the unique value of a such that $f(a) = a$.

(a) $a = 0$

(b) $a = 1$

(c) $a = 3$

(d) $a = 5$

(e) $a = 7$

17. The sequence $2, 3, 5, 6, 7, 8, 10, \ldots$ contains all positive integers that are not perfect squares. Find the 2022^{nd} term of the sequence.

(a) 2022

(b) $\sqrt{2022}$

(c) 2025

(d) 2116

(e) 2067

18. Two squares of side length 3 in. overlap so that the shared region is a square of side length 1 in. Compute the area of the smallest hexagon that covers the two squares.

(a) 16 in.^{2}

(b) 17 in.^{2}

(c) 18 in.^{2}

(d) 21 in.^{2}

(e) 22 in.^{2}

19. Let $ABCD$ be a convex quadrilateral with $AB = \sqrt{2}$, $CD = 2$, and $BD = 1+\sqrt{3}$. If $\angle ABD = 45°$ and $\angle BDC = 30°$, what is the length of $\overline{AC}$?

(a) 2

(b) $\sqrt{2}$

(c) $2 - \sqrt{2}$

(d) $\sqrt{8}$

(e) $1 - \sqrt{3}$

20. Consider the sequence $a_n = a_{n-1} + a_{n-2} + 2022a_{n-3}$ for $n \geq 3$ where $a_0 = 1$, $a_1 = 2$, and $a_2 = 2$. What is the last digit of a_{2022}?

(a) 0

(b) 2

(c) 4

(d) 6

(e) 8