# CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART I

April 18$^{th}$, 2015

1. You row upstream at 6 miles per hour and return on the same route at 12 miles per hour. What was your average speed in miles per hour for the whole trip?

(a) 7 miles per hour
(b) 8 miles per hour
(c) 9 miles per hour
(d) 10 miles per hour
(e) 11 miles per hour

2. For what value of $n$ is it true that $3^1 \cdot 3^2 \cdot 3^3 \cdots 3^n = 3^{351}$?

(a) $n = 24$
(b) $n = 25$
(c) $n = 26$
(d) $n = 27$
(e) No value of $n$

3. A group of students takes a test and the average score is 74. If one more student had taken the test and scored 50, the average would have been 73.6. How many students took the test?

(a) 55
(b) 57
(c) 59
(d) 61
(e) 63

4. Triangle $ABC$ has $AC = 30, BC = 25$, and $AB = 11$. What is the length of the altitude from $C$ to the extension of $AB$?

(a) 18
(b) 20
(c) 22
(d) 24
(e) 26

5. How many zeros are there at the end of the decimal representation of the number 2015!?

(a) 493
(b) 502
(c) 505
(d) 511
(e) 512

6. Each of Alice, Bob, Carol, and Don took a test. Each of them answered at least one question correctly, and altogether they answered 100 questions correctly. Alice had more correct answers than anyone else. Bob and Carol together answered 65 questions correctly. How many correct answers did Don have?

(a) 2
(b) 1
(c) 5
(d) 4
(e) 6

7. During the lunch break from 12 : 00 to 1 : 00, Bill eats, then checks his messages, then goes to the restroom, then talks to a friend. Each activity after the first takes one-third as much time as the preceding activity. There are no intervening time intervals. At what time did Bill finish checking his messages?

(a) 12 : 48
(b) 12 : 50
(c) 12 : 52
(d) 12 : 54
(e) 12 : 56

8. Six weeks is $n!$ seconds. What is the value of $n$?

(a) 6
(b) 7
(c) 8
(d) 10
(e) 12

9. A room contains 2015 light bulbs with one switch for each bulb. All bulbs are off to begin. A group of 2015 people walk in to the room one at a time and perform the following action:

The first person changes the state of all switches, the second person changes the state of every second switch (switches 2, 4, 6, etc.), the third person changes the state of every third switch (switches 3, 6, 9 etc.), and so on, until the $2015^{th}$ person changes the state of the $2015^{th}$ switch.

After the last person has left the room, how many light bulbs are switched on?

(a) 40
(b) 41
(c) 42
(d) 43
(e) 44

10. What is the minimal number of positive divisors of $x$ if $x > 1$, and $x, x^{7/8}$, and $x^{11/12}$ are all integers?

(a) 23
(b) 25
(c) 30
(d) 27
(e) 32

11. What is the number of integers $n$ for which $\frac{9n+45}{n-3}$ is an integer?

(a) 6
(b) 9
(c) 12
(d) 18
(e) 24

12. A quadratic polynomial $f$ satisfies $f(x) \geq 2$ for all $x, f(2) = 2$, and $f(3) = 4$. What is $f(5)$?

(a) 15
(b) 16
(c) 18
(d) 20
(e) 22

13. Tom, Dick, and Harry were playing tennis. After each game, the loser was replaced by the third person. At the end of the day, Tom had played 27 games, and Dick played 13 games. How many games did Harry play?

(a) 11
(b) 12
(c) 13
(d) 14
(e) 15

14. A “hundred-year flood” is defined as a flood of such magnitude that the probability of its occurrence in any given year is 0.01. What is the probability that a 70-year old man has seen at least one hundred-year flood in his lifetime?

(a) $1 - (0.99)^{70}$
(b) $(0.01)^{70}$
(c) $1 - (0.01)^{70}$
(d) 0.70
(e) 1

15. 2015! is divisible by $2^k$. What is the largest possible value of $k$?

(a) 1006
(b) 1509
(c) 1910
(d) 2005
(e) 2010

16. Points $A, C$, and $D$ lie on a circle. Point $B$ lies outside the circle such that $B, D$, and $C$ are collinear with $D$ between $B$ and $C$, and $BA$ is tangent to the circle. If $AB = 2, AC = 3$, and $BD = 1$, what is the area of triangle $ABC$?

(a) 1
(b) 2
(c) $\frac{3}{4}\sqrt{15}$
(d) $\frac{3}{4}\sqrt{11}$
(e) $2\sqrt{11}$

17. A right circular cone has height equal to radius. What is the ratio of its volume to that of a cube inscribed inside it, with the base of the cube lying on the base of the cone?

(a) $\frac{\pi}{12}(10+\sqrt{2})$
(b) $\frac{\pi}{12}(10+3\sqrt{2})$
(c) $\frac{\pi}{12}(10+5\sqrt{2})$
(d) $\frac{\pi}{12}(10+7\sqrt{2})$
(e) $\frac{\pi}{12}(10+9\sqrt{2})$

18. $ABCD$ is a rectangle in which the length $AB$ minus the length $AD$ equals 10. Inside $ABCD$ is a square $WXYZ$ with sides parallel to those of the rectangle, and $W$ closest to $A$, and $X$ closest to $B$. The total of the areas of the trapezoids $XBCY$ and $AWZD$ is 1000, while the total area of the trapezoids $ABXW$ and $ZYCD$ is 400. What is the area of the square $WXYZ$?

(a) 400
(b) 1600
(c) 3600
(d) 4900
(e) 6400

19. How many 10-digit strings of 0's and 1's are there that do not contain any consecutive 0's?

(a) 144
(b) 140
(c) 89
(d) 85
(e) 133

20. Let $BE$ be a median of triangle $ABC$, and let $D$ be a point on $AB$ such that $BD/DA = 3/7$. What is the ratio of the area of triangle $BED$ to that of triangle $ABC$?

(a) 3/20
(b) 7/20
(c) 1/5
(d) 1/4
(e) the answer cannot be detemined

21. You write five letters to different people, and address the corresponding envelopes. In how many ways can the letters be placed in the envelopes, with one letter in each envelope, so that none of them is in the correct envelope?

(a) 36
(b) 40
(c) 42
(d) 44
(e) 52

22. An equilateral triangle in the first quadrant has vertices at the points $(0, 0), (x_1, 4)$, and $(x_2, 11)$. What is the ordered pair $(x_1, x_2)$?

(a) $(\sqrt{3}, 6\sqrt{3})$
(b) $(\sqrt{3}, \sqrt{3})$
(c) $(6\sqrt{3}, \sqrt{3})$
(d) $(6\sqrt{3}, 6\sqrt{3})$
(e) $(3\sqrt{6}, \sqrt{6})$

23. The diagonals of a parallelogram partition it into four triangles. Let $G$ be the centroid of one of the triangles, and let $T$ be a triangle formed by $G$ and two vertices of the parallelogram. What is the largest possible ratio of the area of $T$ to that of the parallelogram?

(a) 3/4
(b) 2/3
(c) 7/12
(d) 1/2
(e) 5/12

24. How many ordered 4-tuples of non-negative integers $(a, b, c, d)$ satisfy $a + b + c + d \leq 15$?

(a) 3524
(b) 3672
(c) 3716
(d) 3876
(e) none of the above

25. Out of all relatively prime integers $a$ and $b$, what is the largest possible value of the greatest common divisor of $a + 201b$ and $201a + b$?

(a) 37542
(b) 39264
(c) 40400
(d) 42176
(e) 44862

26. Each of two urns contains $N$ balls. All balls are either red or black, and each urn contains at least one red ball and one black ball. You randomly select and urn and a ball from it, and then put the ball back. Then you do this again. What is the smallest value of $N$ for which it is possible that the probability that you chose two red balls from the first urn equals the probability that you chose two red balls or two black balls from the second urn?

(a) 9
(b) 4
(c) 12
(d) 6
(e) 7

27. Let $f(x) = x^2 + 10x + 20$. For what real values of $x$ is $f(f(f(f(x)))) = 0$?

(a) $\pm 5^{1/4} - 5$
(b) $\pm 5^{1/8} - 5$
(c) $\pm 5^{1/10} - 5$
(d) $\pm 5^{1/12} - 5$
(e) $\pm 5^{1/16} - 5$

28. In the standard version of tic-tac-toe, two players $X$ and $O$ alternately fill in a $3 \times 3$ board with their symbols, with $X$ moving first. The game stops when one of the two players has three of his/her symbols in the same row, column, or diagonal, or all squares are filled. How many ways of filling in the entire board with $X$'s and $O$'s can be the end result of a valid game of tic-tac-toe?

(a) 78
(b) 82
(c) 86
(d) 100
(e) 122.

29. Find the sum of all positive integers $n$ with no more than 3 digits for which the number obtained as the last three digits of $n^2$ equals $n$.

(a) 973
(b) 986
(c) 1002
(d) 1012
(e) 1013

30. What is the value of $xy + yz + zx$ if $x^2 + xy + y^2 = 2, y^2 + yz + z^2 = 1$, and $z^2 + zx + x^2 = 3$, with $x, y$, and $z$ all positive?

(a) $\frac{2}{3}\sqrt{2}$
(b) $\frac{2}{3}\sqrt{6}$
(c) $\frac{2}{3}\sqrt{7}$
(d) $\sqrt{3}$
(e) none of the above

## Solutions

1. B
2. C
3. C
4. D
5. B
6. B
7. D
8. D
9. E
10. B
11. E
12. D
13. D
14. A
15. D
16. C

17. D
18. C
19. A
20. A
21. D
22. C
23. E
24. D
25. C
26. E
27. E
28. A
29. C
30. B

# CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART II

April 18$^{th}$, 2015

1. Evaluate $\ln(\tan 1^\circ) + \ln(\tan 2^\circ) + \cdots + \ln(\tan 89^\circ)$.

(a) 0
(b) 1
(c) 2
(d) $\ln \pi$
(e) $e$

2. How many statements on the card below are false?

Exactly 4 statements on this card are false.
Exactly 3 statements on this card are false.
Exactly 2 statements on this card are false.
Exactly 1 statements on this card are false.

(a) 4
(b) 3
(c) 2
(d) 1
(e) cannot be determined

3. Simplify

$$\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)} + \frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}$$
$$+ \frac{(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)} + \frac{(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)}$$

Here, $a, b, c, d$ are different real numbers.

(a) 0

(b) $\dfrac{x^3}{(a-b)(b-c)(c-d)(d-a)}$

(c) $\dfrac{(x-a)(x-b)(x-c)(x-d)}{(a-b)(b-c)(c-d)(d-a)}$

(d) 1

(e) 2

4. If $(x^2 + 2xy + y^2)^7$ is expanded completely, then the sum of the numerical coefficients is:

(a) 0
(b) $2^{14} - 1$
(c) $2^7$
(d) $2^7 - 1$
(e) $2^{14}$

5. In a group of five friends, the sums of the ages of each group of four of them are $124, 128, 130, 136$, and $142$ respectively. What is the age of the youngest of the friends?

(a) 18
(b) 21
(c) 23
(d) 25
(e) 34

6. Suppose that $f(x)$ is defined for all values $x > 0$ and suppose that $f(x) + f(1/x) = 0$ for all $x > 0$. What is $f(2)$?

(a) 0
(b) 1
(c) 2
(d) 3
(e) cannot be determined

7. On Saturday, four trucks drove in a straight line (one directly behind the other). None of the trucks passed any of the others, so the order of the trucks never changed. How many ways are there to rearrange the order of the trucks so that on Sunday, no truck is directly behind a truck that it was directly behind on Saturday?

(a) 4
(b) 8
(c) 11
(d) 12
(e) 16

8. The set $\left\{ \frac{z-1}{z+1} \;\middle|\; z \in \mathbb{C},\ |z| < 1 \right\}$ is:

(a) a circle
(b) the entire complex plane
(c) the open left half of the complex plane
(d) the open right half of the complex plane
(e) the complex plane except for the real axis

9. What is the value of the sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{99}{100!}?$$

(a) $\frac{1}{2} - \frac{1}{100!}$

(b) $1 - \frac{1}{100!}$

(c) $1 - \frac{1}{99!}$

(d) $\frac{1}{2} - \frac{1}{99!}$

(e) 1

10. Each edge of a rectangular solid is increased by 50%. The percent increase in the surface area is:

(a) 50%
(b) 100%
(c) 125%
(d) 237.5%
(e) 300%

11. What is the remainder when 100101102103104105106107108 is divided by 999?

(a) 0
(b) 27
(c) 522
(d) 936
(e) 990

12. If $p$ is the perimeter of an equilateral triangle inscribed in a circle, then the area of the circle is:

(a) $\dfrac{\pi p^2}{3}$

(b) $\dfrac{\pi p^2}{9}$

(c) $\dfrac{\pi p^2}{27}$

(d) $\dfrac{\pi p^2}{81}$

(e) $\dfrac{\pi p^2 \sqrt{3}}{27}$

13. How many polynomials $p(x)$ satisfy both $p(12) = 12!$ and $xp(x-1) = (x-12)p(x)$?

(a) 0
(b) 1
(c) 2
(d) 12
(e) infinitely many

14. Let $x = \frac{1}{2}(\sin^{-1}(3/5) + \sin^{-1}(5/13))$. What is the value of $\tan x$?

(a) $\frac{7}{12}$

(b) $\frac{4}{7}$

(c) $\frac{7}{4}$

(d) $\frac{5}{8}$

(e) $\frac{8}{5}$

15. How many integers between 1 and 2015 have exactly 27 positive divisors?

(a) 0
(b) 1
(c) 2
(d) 27
(e) 28

16. Find the value of $\displaystyle\sum_{i=1}^{13} \frac{1}{i^2} - \sum_{i=2}^{14} \frac{1}{i^2}$

(a) $\dfrac{1}{196}$

(b) $\dfrac{195}{196}$

(c) $\dfrac{197}{196}$

(d) $\dfrac{13}{14}$

(e) 1

17. What is the coefficient of $x^{50}$ in the expansion of the following product?

$$(1 + 2x + 3x^2 + 4x^3 + \cdots + 101x^{100}) \cdot (1 + x + x^2 + x^3 + \cdots + x^{25})$$

(a) 50
(b) 125
(c) 501
(d) 923
(e) 1001

18. What is the largest possible value of the function $5\sin x + 12\cos x$?

(a) 12
(b) 13
(c) 17
(d) 14
(e) $\sqrt{119}$

19. Let $P_n = 1^n + 2^n + 3^n + 4^n$. Find the number of integers $n$ for which $1 \leq n \leq 100$ and $P_n$ is a multiple of 5.

(a) 68
(b) 75
(c) 86
(d) 98
(e) 100

20. An equilateral triangle $ABC$ has area 1. Erect an exterior square on each side of $ABC$. The six vertices of these squares which don't coincide with vertices of $ABC$ form a hexagon. What is the area of this hexagon?

(a) $6\sqrt{3}$
(b) $4\sqrt{3}$
(c) $4 + 2\sqrt{3}$
(d) $4(1 + \sqrt{3})$
(e) $10\sqrt{3}/3$

21. Let $x, y$ be real numbers with $x + y = 1$. If $(x^2 + y^2)(x^3 + y^3) = 12$, what is the value of $x^2 + y^2$?

(a) $\sqrt{2}$
(b) $\sqrt{3}$
(c) 2
(d) 3
(e) 4

22. Bubba's summer job is to wash all the folding chairs at the racetrack. There are over 1000 chairs! He decides that in any given week, he can wash just slightly more than half of the remaining chairs. To be precise, suppose there are $n$ chairs remaining at the start of the week. If $n$ is even, he washes $\frac{n}{2}+1$ chairs that week. If $n$ is odd, he washes $\frac{n+1}{2}$ chairs that week. To his surprise he finds that he always has an even number of chairs to wash at the start of each week. What is the smallest number of chairs he could have washed during the summer?

(a) 1020
(b) 1022
(c) 1024
(d) 1026
(e) 1028

23. What is the length of the shortest path that begins at the point $(2, 5)$, touches the $x$-axis and then ends at a point on the circle $(x + 6)^2 + (y - 10)^2 = 16$?

(a) 12
(b) 13
(c) $4\sqrt{10}$
(d) $6\sqrt{5}$
(e) $4 + \sqrt{89}$

24. Find a real number $r$ such that the equation $|x+12|+|x-5|=r$ has infinitely many solutions in $x$.

(a) 5
(b) 7
(c) 12
(d) 17
(e) there is no such real number.

25. Two points $A$ and $B$ lie on a sphere of radius 12. The length of the straight line segment joining $A$ and $B$ is $12\sqrt{3}$. What is the length of the shortest path from $A$ to $B$ if every point of the path is on the sphere?

(a) $6\pi$
(b) $8\pi$
(c) $9\pi$
(d) $10\pi$
(e) $12\pi$

26. Which of the following is largest (all angles are in degrees)?

(a) $\sin(45^\circ) + \cos(45^\circ)$
(b) $\sin(60^\circ) + \cos(60^\circ)$
(c) $\sin(90^\circ) + \cos(90^\circ)$
(d) $\sin(120^\circ) + \cos(120^\circ)$
(e) $\sin(135^\circ) + \cos(135^\circ)$

27. Given that $29a031 \times 342 = 100900b02$ where $a, b$ denote missing digits, what is the value of $a + b$?

(a) 7
(b) 8
(c) 9
(d) 10
(e) 11

28. A person rowing a boat against a river current got from point A to point B in 4 hours. It took him 2 hours to get from point B to point A rowing along the current. How long would it take him to row the same distance as between A and B on a lake with no current?

(a) 2h 30 min
(b) 2h 40 min
(c) 3h
(d) 3h 15 min
(e) 3h 20 min

29. A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the $x$-intercepts of these lines?

(a) 4
(b) 6
(c) 8
(d) 10
(e) 12

30. What is the number of ordered pairs of nonnegative integers $(x, y)$ which satisfy the equation
$\sqrt{x} + \sqrt{y} = \sqrt{2015}$?

(a) None
(b) 2
(c) 4
(d) 6
(e) 8

## Solutions

1. A
2. B
3. D
4. E
5. C
6. E
7. C
8. C
9. B
10. C
11. D
12. C
13. B
14. B
15. C
16. B

17. E
18. B
19. B
20. D
21. D
22. B
23. B
24. D
25. B
26. A
27. E
28. B
29. D
30. B