2015
Leap Frog Relay Grades 9-10
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. The number 2015 has how many *prime* factors? (Note that 1 is *not* a prime number.)

   (a) 2 (b) 3

   (c) 4 (d) 5

   (e) None of these

   Solution. (b) Factor 2015 as $5 \times 13 \times 31$ to get 3 prime factors.

2. $1 - 2 + 3 - 4 + 5 - 6 + \cdots - 2014 + 2015 =$ ________

   (a) 1004 (b) 1006

   (c) 1008 (d) 1010

   (e) None of these

   Solution. (c)

$$\begin{aligned} 1 - 2 + 3 - 4 + 5 - 6 + \cdots - 2014 + 2015 &= \underbrace{(-1) + (-1) + \cdots + (-1)}_{1007} + 2015 \\ &= -1007 + 2015 \\ &= 1008. \end{aligned}$$

3. Assuming you can trade 3 apples for 7 oranges and 12 oranges for 5 kiwi, how many apples can you get for 35 kiwi?

(a) 42 Apples

(b) 40 Apples

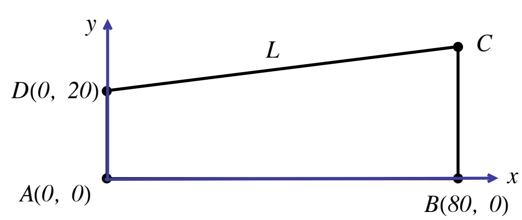
(c) 38 Apples

(d) 36 Apples

(e) None of these

Solution. **(d)** One apple is worth (7/3) oranges and one orange is worth (5/12) of a kiwi. Thus, one apple is worth (7/3)(5/12) = (35/36) of a kiwi. This tells us that 35 kiwi are worth 36 apples.

4. What is the slope of the line L if the area enclosed by the trapezoid $ABCD$ is equal to 2015?



(a) Slope of $L = \dfrac{83}{640}$

(b) Slope of $L = \dfrac{83}{720}$

(c) Slope of $L = \dfrac{93}{640}$

(d) Slope of $L = \dfrac{93}{720}$

(e) None of these

Solution. **(a)** Write the coordinates of point C as $(80, y)$. Then the area of the trapezoid $ABCD$ is $(1/2)(20 + y) \times 80$. This gives us the equation to solve,

$$\frac{1}{2}(20 + y) \times 80 = 2015.$$

Solving, we get, $y = 243/8$. The slope of L is then

$$\frac{243/8 - 20}{80} = \frac{83}{640}$$

5. Suppose that when dividing the number n by 7, there results a remainder of 3. What then is the remainder if you were to divide the number $2015n$ by 7?

   (a) 0 (b) 1

   (c) 2 (d) 3

   (e) None of these

   Solution. **(e)** Since dividing n by 7 results in a remainder of 3, we have $n = 7q + 3$ for some integer $n \geq 0$. Thus,

   $$2015n = 2015 \cdot 7q + 2015 \cdot 3 = 7(2015q) + 6045.$$

   But, then dividing 6045 by 7 results in a quotient of 863 with a remainder of 4,

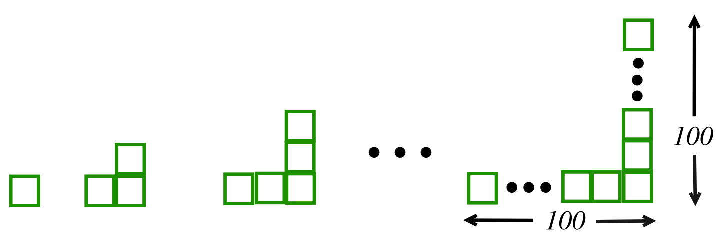
   $$6045 = 7 \cdot 863 + 4.$$

   Thus,

   $$\begin{aligned} 2015n &= 7(2015q) + 7 \cdot 864 + 4 \\ &= 7(2015q + 864) + 4 \\ &= 7q' + 4, \end{aligned}$$

   where $q' = 2015q + 4$. This means that dividing $2015n$ by 7 results in a remainder of 4, none of the answer choices presented.

6. In the figure below, each shape is made up of 1×1 squares. Assuming the pattern continues for the indicated 100 shapes, determine the *total* number of 1×1 squares used for the 100 shapes.



(a) 5,050 squares

(b) 10,000 squares

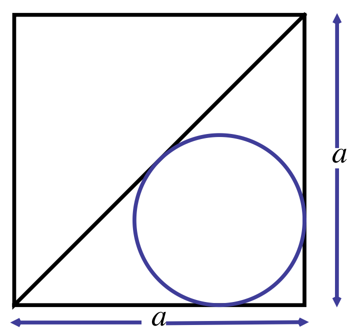
(c) 20,100 squares

(d) 10,100 squares

(e) None of these

Solution. **(b)** Note that each shape fits neatly into the crook of the next shape. So, for example, the first 3 shapes fit together in a 3×3 square, using $3^2 = 9$ blocks. So, the 100 shapes will fit together in a 100×100 square, using $100^2 = 10,000$ squares.

7. In the figure below, the rectangle is a square, whose side lengths are all equal to the value a, and the circle is inscribed as pictured. Determine the radius, r, of the inscribed circle.



(a) $r = a(\sqrt{2}/2)$

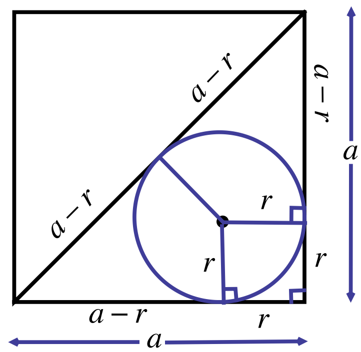
(b) $r = a(1 - \sqrt{2}/2)$

(c) $r = a(\sqrt{2} - 1)$

(d) $r = a(2 - \sqrt{2})$

(e) None of these

Solution. **(b)** Label the figure as follows.



Notice that the diagonal has length $\sqrt{2}a$. And so, we get the equation $\sqrt{2}a = 2(a - r)$. Solving for r gives us

$$r = a(1 - \sqrt{2}/2).$$

8. *For how many* of the ten digits $x = 0, 1, 2, \ldots, 9$ is the 2017-digit number $n = 1\underbrace{xx\ldots x}_{2015}0$ divisible by 24?

(a) 0

(b) 1

(c) 2

(d) 3

(e) None of these

Solution. **(b)** Since $24 = 8 \times 3$, we must have that n is divisible by 3 and 8. A number is divisible by 3 when its digit sum is divisible by 3. And so we must have that $1 + 2015x$ is divisible by 3. Now, $2015 = 2013+2$, and so $1+2015x = (1+2x)+2013x = (1+2x)+3(671x)$. Thus, $1 + 2015x$ divisible by 3 when $1 + 2x$ is divisible by 3. The possible vales for x to insure that n is divisible by 3 are then reduced to $x = 1, 4, 7$.

Finally, n is divisible by 8 precisely when the 3-digit number $xx0$ is divisible by 8. It is an easy check that of the digits $x = 1, 4, 7$, only $x = 4$ satisfies this condition.

So there is one possible solution, $x = 4$.

9. If the vertex of the parabola $y = ax^2 + bx + c$ lies on the x-axis, then ________.

(a) $b^2 = 4ac$

(b) $c = 0$

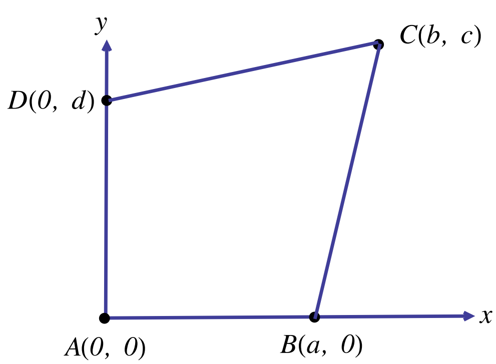
(c) $a + b + c = 0$

(d) $b = 0$

(e) None of these

Solution. **(a)** If the vertex of the parabola lies on the x-axis, then this means there is only one real solution, x, to the equation $ax^2 + bx + c = 0$. The solutions given by the quadratic formula are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$. It follows that if there is only one solution, then $\sqrt{b^2 - 4ac}$ must be equal to zero. Thus, $b^2 = 4ac$.

10. Quadrilateral $ABCD$ in the cartesian plane is pictured below. determine the area enclosed by $ABCD$. (You may assume $b > a$ and $c > d$ as pictured.)



(a) Area $= \frac{1}{4}(a + b)(d + c)$

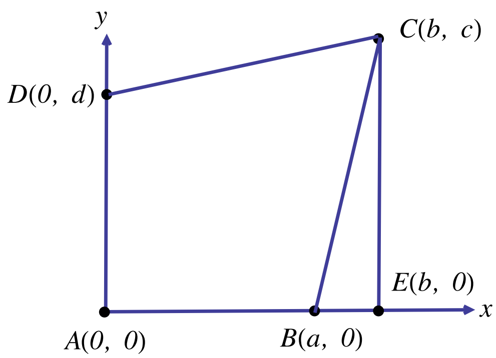
(b) Area $= \frac{1}{4}(a + d)(b + c)$

(c) Area $= \frac{1}{2}(ad + bc)$

(d) Area $= \frac{1}{2}(ac + bd)$

(e) None of these

Solution. **(d)** Drop a perpendicular segment $\overline{CE}$ from point C to the x-axis as pictured below.



The area enclosed by $ABCD$ is the difference of the trapezoid area $AECD$ and the triangle area BEC.

$$\begin{aligned}\text{Area}(ABCD) &= \text{Area}(AECD) - \text{Area}(BEC) \\ &= \frac{1}{2}(c+d)b - \frac{1}{2}(b-a)c \\ &= \frac{1}{2}(ac+bd).\end{aligned}$$

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Part II Solutions

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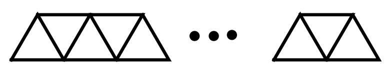
11. Sticks are placed on a table to form a connected string of triangles as pictured below. Assuming each triangle side consists of a single stick and there are 2015 triangles, how many sticks are used?



(a) 4029 (b) 4030

(c) 4031 (d) 4032

(e) None of these

Solution. (c) After the first triangle, which consists of 3 sticks, each additional triangle is gotten be adding 2 sticks the the figure. Thus, the total number of sticks used is

$$3 + 2014(2) = 4031.$$

12. You are on a road trip. For the first 10 miles you average 50 miles per hour. For the next 8 miles, you average 60 miles per hour. And for the last 4 miles you average 40 miles per hour. How many minutes were you traveling?

(a) 20 minutes

(b) 26 minutes

(c) 32 minutes

(d) 38 minutes

(e) None of these

Solution. **(b)** Use the formula Time = Distance/Rate. During the first leg of your journey, you travel for $10/50 = 1/5$ of an hour, which is 12 minutes. For the second leg, you travel 8/60 of an hour, which is 8 minutes. Finally, the last leg takes $4/40 = 1/10$ of an hour, which is 6 minutes. Adding these minutes gives a total of $12 + 8 + 6 = 26$ minutes.

13. How many positive integer divisors of 1,000,000 are there?

(a) 49

(b) 50

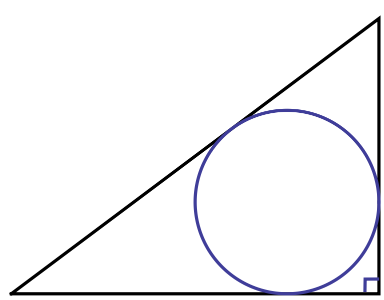
(c) 36

(d) 100

(e) None of these

Solution. **(a)** Write 1,000,000 as $10^6 = 2^6 \cdot 5^6$. The positive integer divisors are then in the form $2^x \cdot 5^y$ where $x = 0, 1, \ldots, 6$ and $y = 0, 1, \ldots, 6$. There are 7 possible solutions for x and 7 possible solutions for y, giving $7 \cdot 7 = 49$ positive divisors.

14. What is the radius of the inscribed circle of a 3-4-5 right triangle?



(a) radius $= \frac{1}{\sqrt{2}}$

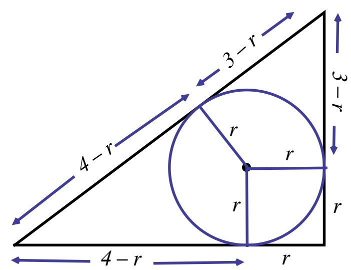
(b) radius $= \frac{2}{1+\sqrt{2}}$

(c) radius $= \sqrt{2}$

(d) radius $= 2\sqrt{2} - 2$

(e) None of these

Solution. **(e)** Let r represent the radius of the inscribed circle. We use the fact that the two tangent segments to a circle from an external point are congruent, allowing us to label the figure as pictured.



Note that the hypotenuse of the right triangle has length equal to 5, giving us

$$(4 - r) + (3 - r) = 5 \Longrightarrow r = 1,$$

none of the answer choices provided.

15. $\dfrac{1}{20+\sqrt{15}} + \dfrac{1}{20-\sqrt{15}} =$ ______

(a) $\dfrac{8}{77}$

(b) $\dfrac{9}{77}$

(c) $\dfrac{10}{77}$

(d) $\dfrac{12}{77}$

(e) None of these

Solution. **(a)** We will rationalize the denominators.

$$\begin{aligned} \frac{1}{20+\sqrt{15}} &= \frac{1}{20+\sqrt{15}} \cdot \frac{20-\sqrt{15}}{20-\sqrt{15}} \\ &= \frac{20-\sqrt{15}}{400-15} \end{aligned}$$

$$= \frac{20-\sqrt{15}}{385}.$$

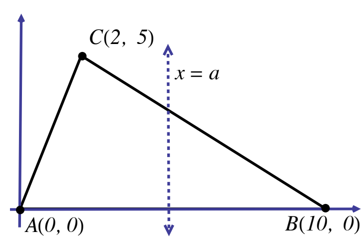
And,

$$\begin{aligned}
\frac{1}{20-\sqrt{15}} &= \frac{1}{20-\sqrt{15}} \cdot \frac{20+\sqrt{15}}{20+\sqrt{15}} \\
&= \frac{20+\sqrt{15}}{400-15} \\
&= \frac{20+\sqrt{15}}{385}.
\end{aligned}$$

Now, we add,

$$\begin{aligned}
\frac{1}{20+\sqrt{15}} + \frac{1}{20-\sqrt{15}} &= \frac{20-\sqrt{15}}{385} + \frac{20+\sqrt{15}}{385} \\
&= \frac{40}{385} \\
&= \frac{8}{77}.
\end{aligned}$$

16. What is the value of a so that the vertical line $x = a$ divides the triangle $\triangle ABC$ pictured below into two regions of equal area?



(a) $a = \sqrt{7}$

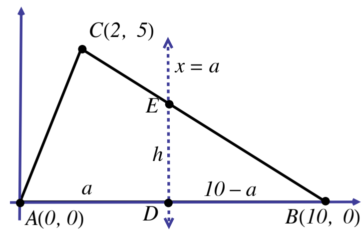
(b) $a = \frac{7}{2}$

(c) $a = 3$

(d) $a = 10 - 2\sqrt{10}$

(e) None of these

Solution. **(d)** First, we note that the area of $\triangle ABC$ is 25. Now, label the points D and E as pictured below. Let $h = DE$, the height of $\triangle DBE$.



The slope of $\overleftrightarrow{BC}$ is

$$\text{Slope}\,\overleftrightarrow{BC} = \frac{5}{2-10} = -\frac{5}{8}.$$

And so we can solve for h from the equation

$$-\frac{h}{10-a} = -\frac{5}{8} \Longrightarrow h = \frac{5}{8}(10-a).$$

Thus, the area of $\triangle DBE$ is $\frac{1}{2}(10-a)\cdot\frac{5}{8}(10-a) = \frac{5}{16}(10-a)^2$. We then solve for a from the equation

$$\frac{5}{16}(10-a)^2 = \frac{25}{2}.$$

There are two solutions, $a = 10 \pm 2\sqrt{10}$. We ignore the + solution since a must be less than 10. Thus we get

$$a = 10 - 2\sqrt{10}.$$

17. A 20% price decrease, followed by a 20% price increase is equivalent to ________.

(a) A 4% price increase. (b) A 4% price decrease.

(c) A 2% price decrease. (d) A 2% price increase.

(e) None of these

Solution. **(b)** A 20% price decrease followed by a 20% price increase changes the price by a factor $(.8)(1.2) = .96$, which is a 4% price decrease.

18. What is the equation of the line with positive slope that goes through the origin and is tangent to the circle $(x-4)^2 + y^2 = 4$?

(a) $y = x/\sqrt{11}$

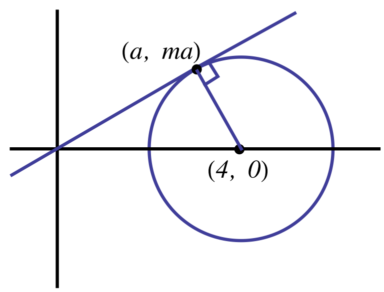
(b) $y = x/\sqrt{7}$

(c) $y = x/\sqrt{5}$

(d) $y = x/\sqrt{3}$

(e) None of these

Solution. **(d)** Write the line through the origin as $y = mx$ and the point of tangency as (a, ma).



First we note that the line is perpendicular to the radius, which gives us

$$\frac{ma}{a-4} = -\frac{1}{m} \implies m^2 = \frac{4-a}{a}.$$

Secondly, the point (a, ma) lies on the circle, and so the equation is satisfied:

$$(a-4)^2 + m^2a^2 = 4.$$

Substitute $m^2 = (4-a)/a$ into the above equation and solve,

$$\begin{aligned}(a-4)^2 + \left(\frac{4-a}{a}\right)a^2 = 4 \quad &\implies \quad a^2 - 8a + 16 + 4a - a^2 = 4 \\ &\implies \quad a = 3.\end{aligned}$$

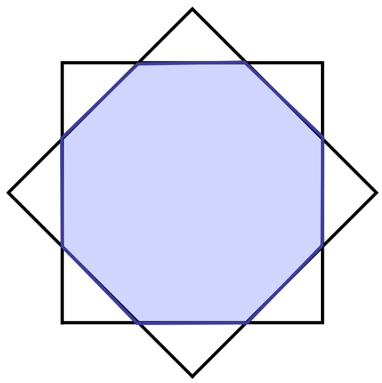
From this, we get

$$\begin{aligned}m^2 = \frac{4-3}{3} \quad &\implies \quad m^2 = \frac{1}{3} \\ &\implies \quad m = \pm\frac{1}{\sqrt{3}}.\end{aligned}$$

We take the positive slope, giving us the line

$$y = x/\sqrt{3}.$$

19. Two $2' \times 2'$ squares share the same center and one square is rotated $45°$ with respect to the other square (see picture below). Determine the shaded area that is enclosed by both squares.



(a) Shaded Area $= 4\sqrt{2} - 4$ ft^2.

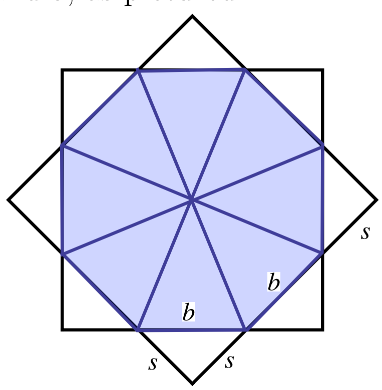
(b) Shaded Area $= 4\sqrt{2} + 4$ ft^2.

(c) Shaded Area $= 2\sqrt{2} + 2$ ft^2.

(d) Shaded Area $= 8\sqrt{2} - 8$ ft^2.

(e) None of these

Solution. **(d)** We first note that the shaded octagon is a regular octagon due to its rotational symmetry. We can then divide the shaded octagon into 8 isosceles triangles, as pictured below. We have also labeled lengths b and s, as pictured.



This gives us two equations, $b^2 = 2s^2$ (from the Pythagorean Theorem) and $b + 2s = 2$, since the side length of a square is equal to 2. Substitute $b = \sqrt{2}s$ into the second equation to get $s = 2/(2+\sqrt{2})$. Substitute this back into the equation $b = \sqrt{2}s$ to get

$$\begin{aligned} b &= \frac{2\sqrt{2}}{2+\sqrt{2}} \\ &= \frac{2\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} \\ &= 2\sqrt{2} - 2. \end{aligned}$$

The area of one of the eight shaded isosceles triangles is then equal to $(1/2) \cdot 1 \cdot (2\sqrt{2} - 2) = \sqrt{2} - 1$. Thus, the total shaded area is $8\sqrt{2} - 8$ ft^2

20. The number $\sqrt{20+\sqrt{15}}$ is the root of a degree 4 polynomial $p(x) = x^4 + bx^3 + cx^2 + dx + e$ with integer coefficients. That is, b, c, d and e are all integers. Determine the value of $p(1)$.

(a) 346 (b) 348

(c) 350 (d) 352

(e) None of these

Solution. **(a)** Let $x = \sqrt{20+\sqrt{15}}$. First square x,

$$x^2 = 20 + \sqrt{15}.$$

Then, subtract 20 from each side and square again.

$$(x^2 - 20)^2 = 15 \Longrightarrow x^4 - 40x^2 + 400 = 15 \Longrightarrow x^4 - 40x^2 + 385.$$

This tells us that the number $\sqrt{20+\sqrt{15}}$ is a root of the polynomial

$$p(x) = x^4 - 40x^2 + 385.$$

Thus,

$$\begin{aligned} p(1) &= 1 - 40 + 385 \\ &= 346. \end{aligned}$$