Mad Hatter 11-12. Part I.

Math Field Day. California State University, Fresno.
April 16^{th}, 2011.

Question 1.

When a triangle's base is increased by 10%, and the altitude to this base is decreased by 10%, the change in area is

(a) 1% increase
(b) 1% decrease
(c) 0% change
(d) There is no way to determine from the given information.
(e) None of the above.

Question 2.

A cubic equation $x^3 - 4x^2 - 11x + a = 0$ has roots x_1, x_2, x_3. If $x_1 = x_2 + x_3$, what is a?

(a) 24
(b) 28
(c) 30
(d) 32
(e) 36

Question 3.

The value of $\dfrac{(1-\sqrt{5})(2+\sqrt{5})}{7+\sqrt{5}}$ is:

(a) $\dfrac{1+\sqrt{5}}{4}$
(b) $\dfrac{4-\sqrt{5}}{11}$
(c) $\dfrac{-4-\sqrt{5}}{11}$
(d) $\dfrac{8+4\sqrt{5}}{11}$
(e) None of the above.

Question 4.

For which values of a is the polynomial $P(x) = x^{1000} + ax + 9$ divisible by $x + 1$?

(a) None.
(b) 10
(c) -10
(d) 9
(e) -9

Question 5.

Suppose you visit Venus and meet some aliens who teach you their system of counting. You notice that they use a true place value system that is similar to ours, but the Venusians use base 6 instead of base 10. The table below shows how to translate their characters into our digits.

#	&	<	@	/	*
0	1	2	3	4	5

Table: Venusian Characters

Convert the number 54 (base 10) to its Venusian equivalent.

(a) & @
(b) & @ #
(c) @ &
(d) #@&
(e) * /

Question 6.

Let $x = \left(\frac{1}{2}\right)^{\left(\frac{1}{2}^{\left(-\frac{1}{2}\right)}\right)}$. To which of the following intervals does x belong?

(a) $(0, 1/8]$
(b) $(1/8, 1/4]$
(c) $(1/4, 1/2]$
(d) $(1/2, 1]$
(e) $(1, \infty)$

Question 7.

A room is shaped like a cube with sides of length 4 meters. Suppose that A and B denote two corners of the room that are farthest from each other. A caterpillar is crawling from A to B along the walls. What is the shortest possible length of the trip?

(a) $4\sqrt{2}$ meters
(b) $4\sqrt{3}$ meters
(c) $4\sqrt{5}$ meters
(d) 8 meters
(e) $4\sqrt{6}$ meters

Question 8.

A plane passes through the center of a cube and is perpendicular to one of the cubes diagonals. How many edges of the cube does the plane intersect? (An edge is a line connecting adjacent corners.)

(a) 3
(b) 4
(c) 5
(d) 6
(e) 8

Question 9.

Four numbers are written in a row. The average of the first two numbers is 5. The average of the middle two numbers is 4 and the average of the last two numbers is 10. What is the average of the first and last numbers?

(a) 9
(b) 10
(c) 10.5
(d) 11
(e) 11.5

Question 10.

The number of values of x which satisfy the equation $\frac{2x^2 - 10x}{x^2 - 5x} = x - 3$ is:

(a) One
(b) Three
(c) Two
(d) Zero
(e) None of the above.

Question 11.

The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Squares containing numbers do not contain mines. Each square that does not have a number either has a single mine or nothing at all. How many mines are there?

	1		2		2
1		2		2	
	3		2		1
2		4		1	
	3		3		1
1		3		2	

(a) 9 (b) 10 (c) 11 (d) 12 (e) 13

Question 12.

John and Bill toss a biased coin that has a 60% chance of coming up heads and a 40% chance of coming up tails. They flip the coin until either two heads or two tails in a row are observed. Bill is a winner if two heads in a row are observed first. What is the probability that Bill will win?

(a) 61/95
(b) 63/95
(c) 64/95
(d) 67/95
(e) 69/95

Question 13.

How many times in a 24 hour period do the hour and minute hands of a clock form a right angle?

(a) 48
(b) 44
(c) 34
(d) 24
(e) 22

Question 14.

A pyramid has a 10 ft. by 10 ft. square base. The height of the pyramid is also 10 ft. If we want to slice the pyramid into two pieces with equal volumes by cutting it with a plane parallel to the base, how far above the base should we make the cut?

(a) 5 ft
(b) $5 - \sqrt[3]{4}$ ft
(c) $10 - 5\sqrt[3]{4}$ ft
(d) $\frac{10}{3}$ ft
(e) None of the above.

Question 15.

How many pairs (x, y) of positive integers satisfy $2x + 7y = 1000$?

(a) 70
(b) 71
(c) 72
(d) 73
(e) 74

Question 16.

Let A be the point $(7, 4)$ and D be the point $(5, 3)$. What is the length of the shortest path $ABCD$, where B is a point $(x, 2)$ and C is a point $(x, 0)$? This path consists of three connected segments, with the middle one vertical.

(a) $2 + \sqrt{29}$
(b) $\sqrt{31}$
(c) $2 + \sqrt{31}$
(d) $2 + \sqrt{33}$
(e) $\sqrt{41}$.

Question 17.

When $x^3 + k^2x^2 - 2kx - 6$ is divided by $x + 2$ the remainder is 10, then k must be

(a) -2 or 3
(b) -1
(c) 2 or 3
(d) None of the above.

Question 18.

The number of digits in $2^{12} \times 5^8$ is:

(a) 6
(b) 7
(c) 8
(d) 9
(e) None of the above.

Question 19.

The fourth power of $\sqrt{1 + \sqrt{1 + \sqrt{1}}}$ is:

(a) $3 + 2\sqrt{2}$
(b) 3
(c) $1 + 2\sqrt{3}$
(d) 9
(e) None of the above.

Question 20.

The smallest value of $x^2 + 8x$ for real values of x is:

(a) -16.25
(b) -16
(c) 0
(d) -8
(e) None of the above.

Question 21.

Six people are at a meeting, and at the end of the meeting each person shakes every other person's hand exactly once. How many hand shakes take place?

(a) 30
(b) 15
(c) 5
(d) 3
(e) None of the above.

Question 22.

Let $a, b \in \mathbb{R}$. A student wrote that the product of $a + i$ and $b - i$ was $a + b + i$, where $i^2 = -1$. If this was correct then the minimum value of ab is:

(a) 0
(b) 1
(c) 2
(d) -2
(e) None of the above.

Question 23.

The sum and product of two numbers x and y are each equal to the same positive value. Then the difference $y^2 - x^2$ is:

(a) 0
(b) $\dfrac{(x-1)^2}{x^2}$
(c) $\dfrac{x^2}{(x-1)^2}$
(d) $\dfrac{(2-x)x^3}{(x-1)^2}$
(e) None of the above.

Question 24.

The value of $25^{\log_5 6}$ is:

(a) 36
(b) 6
(c) 5
(d) $\log_5 6$
(e) None of the above.

Question 25.

The distance between the two complex numbers $1-2i$ and $2i-2$ is:

(a) $3-4i$
(b) $-3+4i$
(c) 5
(d) $\sqrt{17}$
(e) None of the above.

Question 26.

If $\dfrac{4^x}{2^{x+y}}=8$ and $\dfrac{9^{x+y}}{3^{5y}}=243$, with x, y real numbers, then xy equals:

(a) 12
(b) 6
(c) 4
(d) $\frac{12}{5}$
(e) None of the above.

Question 27.

Suppose $f(0)=3$ and $f(n)=f(n-1)+2$. Let $T=f(f(f(f(5))))$. What is the sum of the digits of T?

(a) 6
(b) 7
(c) 8
(d) 9
(e) 10.

Question 28.

Four sets $A, B, C,$ and D each have 500 elements. The intersection of any two of the sets has 115 elements. The intersection of any three of the sets has 52 elements. The intersection of all four sets has 30 elements. How many elements are there in the union of the four sets?

(a) 1465
(b) 1472
(c) 1482
(d) 1488
(e) 1512.

Question 29.

How many three digit numbers can be written as a sum of a three digit number and its (one-, two-, or three-digit) reversal?

(a) 75
(b) 80
(c) 85
(d) 90
(e) 95

Question 30.

If (x, y, z) satisfy the three equations below, what is $x + y + z$?

$$\begin{aligned} 2x - y + z &= 7 \\ x + 2y + 3z &= 1 \\ -x + y + 5z &= 6 \end{aligned}$$

(a) -1
(b) 0
(c) 1
(d) 2
(e) 3

Question 31.

How many positive integer divisors does $N = 250 \cdot 88$ have?

(a) 24
(b) 28
(c) 30
(d) 32
(e) 40

Question 32.

A $4 \times 4 \times 4$ wooden cube is painted on five of its faces and is then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that the face showing is painted? Express your answer as a fraction.

(a) 5/24
(b) 1/4
(c) 7/24
(d) 1/3
(e) 1/2

Question 33.

How many integer triples (x, y, z) satisfy both

→ x, y, and z are positive integers less than 30 and

→ $xy^2z^3 = 10,000$?

(a) 5
(b) 6
(c) 7
(d) 8
(e) 9

Question 34.

What is the sum of the three positive integers a, b, and c that satisfy

$$a + \frac{1}{b + \frac{1}{c}} = 5.4?$$

(a) 8
(b) 9
(c) 10
(d) 11
(e) 12

Question 35.

Use each of the digits $2, 3, 4, 6, 7, 8$ exactly once to construct two three-digit numbers M and N so that $M - N$ is positive and is as small as possible. Compute $M - N$.

(a) 19
(b) 29
(c) 39
(d) 49
(e) 59

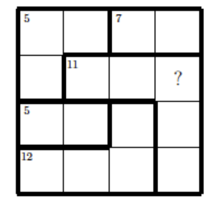
Question 36.

How many integers x with $1 \leq x \leq 100$ satisfy the equation $x^2 + x^3 = y^2$ for some integer y?

(a) 6
(b) 7
(c) 8
(d) 9
(e) 10

Question 37.

Consider the 4×4 puzzle below. The solution uses the numbers 1 to 4 exactly once in each row and each column. The sum of the digits in each cage is the number given in the upper left corner of one of the squares. What digit goes in the square with the question mark in it?



(a) 1
(b) 2
(c) 3
(d) 4
(e) The puzzle has no solution.

Question 38.

A careless teacher with 4 students has 4 report cards, one for each student. The teacher hand delivers the report cards to the students' lockers (here of course each student has his/her own locker). How many ways can the teacher place the report cards in, at least one, incorrect lockers, with one and only one report card being placed in each locker?

(a) 23
(b) 4
(c) $4^4 - 1$
(d) $3! + 1$
(e) None of the above.

Question 39.

A farmer can plow the entire circular field of radius 30 yards, in two hours, and her daughter can plow two-thirds of the same field in two hours. What is the area (in square yards) of the field that can be plowed in an hour if farmer and daughter work together?

(a) 750π
(b) 750
(c) 800π
(d) 800
(e) None of the above.

Question 40.

A circle has center O and points N and P are on the circle. Suppose that $\angle NOP$ is 120°. What is $\angle ONP$?

(a) 30°
(b) 35°
(c) 60°
(d) 70°
(e) 125°

Mad Hatter 11-12. Part II.

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Question 1.

The graphs of the equations $x^2 + y^2 = 1$ and $x + y = 5$ have the following number of intersection points:

(a) Infinitely many
(b) 0
(c) 1
(d) 2
(e) 3

Question 2.

If b people take c days to lay f bricks, then the number of days it will take c people working at the same rate to lay b bricks, is:

(a) fb^2
(b) $\dfrac{b}{f^2}$
(c) $\dfrac{f^2}{b}$
(d) $\dfrac{b^2}{f}$
(e) None of the above.

Question 3.

An object is projected vertically upward from the top of a building with an initial velocity 32 ft/s. Its distance $s(t)$ in feet above the ground is given by the formula $s(t) = -4t^2 + 32t + 25$. What is the object's maximum height above the ground?

(a) 89
(b) 4
(c) 25
(d) 8
(e) None of the above

Question 4.

If one root of $f(x) = x^3 - 2x^2 - 16x + 16k$ is 2, find the other root(s).

(a) 4
(b) 4 and -4
(c) 8
(d) 4 and 8
(e) None of the above

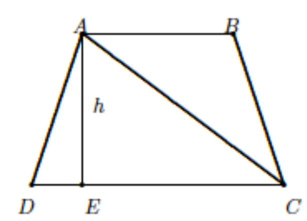
Question 5.

If $\frac{17}{x} = \frac{11}{319}$, then the digit in the tens place of x is:

(a) 3
(b) 9
(c) 1
(d) 5
(e) None of the above

Question 6.

$ABCD$ is an isosceles trapezoid with AB parallel to DC, $AC = DC$, and $AD = BC$. If the height h of the trapezoid is equal to AB, find the ratio $AB : DC$.



(a) $2:3$
(b) $3:5$
(c) $4:5$
(d) $5:7$
(e) $5:9$

Question 7.

If $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$, which of the following is true?

(a) $x = 1$
(b) $1 < x < 2$
(c) $0 < x < 1$
(d) x is infinite.
(e) $x > 2$.

Question 8.

Oscar has a collection of 50 songs that are each 3 minutes in length and 50 songs that are each 5 minutes in length. What is the maximum number of songs from his collection that he can play in 5 hours?

(a) 70
(b) 80
(c) 90
(d) 100
(e) 60

Question 9.

Suppose $a > 0$. If $x^a - \frac{4}{x^{-\frac{a}{2}}} + 4 = 0$, then the value of $y = \frac{1}{x^a + \frac{1}{2}}$ is

(a) 10/45
(b) 45/10
(c) 45/100
(d) 1000/45
(e) None of the above.

Question 10.

When the length of each of the edges of a cube is increased by 1 cm, the cube's total surface area increases by 78 cm^2. What is the length of an edges of the original cube?

(a) 4 cm
(b) 5 cm
(c) 6 cm
(d) 7 cm
(e) None of the above.

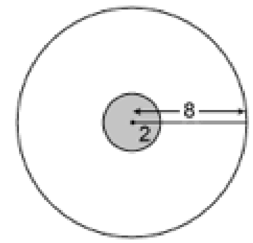
Question 11.

The Fresno State Math Club decides to sell t-shirts to raise money for buying pies for π day. It costs the class \$8 for each t-shirt made. They decide to sell the t-shirts for \$12 each. If the class orders 500 t-shirts to be made, how many t-shirts must they sell before they start to earn a profit?

(a) 250
(b) 333
(c) 334
(d) 750
(e) None of the above

Question 12.

Alex wore a blindfold and shot an arrow at the target shown. Judging by the noise made on impact, he can tell that he hit the target. What is the probability that he hit the shaded region shown?



(a) $\frac{1}{4}$
(b) $\frac{1}{8}$
(c) $\frac{1}{16}$
(d) $\frac{1}{32}$
(e) $\frac{1}{64}$

Question 13.

The sum of the squares of the lengths of the three sides of a right triangle is 200. Then, the length of the hypotenuse is:

(a) 9
(b) $10\sqrt{2}$
(c) 20
(d) 11
(e) None of the above

Question 14.

How many of the numbers from 10 through 100 have the sum of their digits equal to a perfect square?

(a) 18
(b) 16
(c) 10
(d) 14
(e) None of the above

Question 15.

When the last digit of a certain 6-digit number N is transferred to the first position, the other digits moving one place to the right, the new number is exactly one-third of N. The sum of the six digits of N is:

(a) 25
(b) 26
(c) 27
(d) 28

Question 16.

Exactly one of the following numbers is prime. Which is it?

(a) $107^4 - 1$
(b) $6^{33} + 3^{44}$
(c) $2^{61} - 1$
(d) $4^{73} - 1$
(e) $17^{63} - 7^{31}$

Question 17.

The last digit of 2011^{2011} is:

(a) 1
(b) 3
(c) 5
(d) 6
(e) 9

Question 18.

If α, β are solutions of $x^2 + px + q = 0$, then $\alpha^2 + \beta^2$ equals:

(a) $-p^2 - 2q$
(b) $p^2 + q^2$
(c) $p^2 - 2q$
(d) $q^2 - 2p$
(e) $p^2 - 2q^2$

Question 19.

If $x^2 + y^2 = 20$ and $x^2 - y^2 = 2$, what is the value of $|xy|$?

(a) $\frac{3\sqrt{11}}{2}$
(b) 5
(c) $\frac{\sqrt{418}}{2}$
(d) $\frac{5\sqrt{10}}{3}$
(e) $3\sqrt{11}$

Question 20.

If 64 is divided into three parts proportional to 2, 4 and 6, then the smallest part is

(a) 16/3
(b) 11
(c) 32/3
(d) 5
(e) 11/2

Question 21.

What is the largest power of 2 that divides $2^{2011} + 10^{2011}$?

(a) 2^{2000}
(b) 2^{2009}
(c) 2^{2010}
(d) 2^{2011}
(e) 2^{2012}

Question 22.

Evaluate

$$\frac{1 - 2^{-1} + 2^{-2} - 2^{-3} + 2^{-4} - \cdots - 2^{-2009} + 2^{-2010} - 2^{-2011}}{1 + 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + \cdots + 2^{-2009} + 2^{-2010} + 2^{-2011}}$$

(a) 1
(b) 1/2
(c) 1/3
(d) 1/4
(e) 1/5

Question 23.

For what values of b is $\dfrac{2+i}{bi-1}$ a real number? (Here $i^2 = -1$).

(a) -1.5
(b) -0.5
(c) 0.5
(d) 2
(e) 1.5

Question 24.

How many polynomials are there of the form $x^3 - 8x^2 + cx + d$ such that c and d are real numbers and the three roots of the polynomial are distinct positive integers?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

Question 25.

Suppose a, b are positive integers, $a < 10$ and $f(x) = ax + b$, $g(x) = bx + a$. If $f(g(50)) - g(f(50)) = 28$. What is (a, b) ?

(a) $(3, 4)$
(b) $(7, 4)$
(c) $(6, 2)$
(d) $(4, 1)$
(e) $(5, 2)$

Question 26.

Each edge of a cube is increased by 30%. What percentage does the surface area of the cube increase?

(a) 100%
(b) 110%
(c) 120%
(d) 130%
(e) None of these.

Question 27.

How many solutions does the equation $\tan x = x$ have?

(a) No solution
(b) 2
(c) 4
(d) 8
(e) Infinitely many

Question 28.

The remainder of $1^5 + 2^5 + \cdots + 10^5$ upon dividing by 4 is

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

Question 29.

Four suspects of a crime made the following statements to the police:
Ernesto: Paul did it.
Oscar: I did not do it.
Paul: Tom did it.
Tom: Paul lied when he said I did it
If the crime was committed by only one person, and exactly one of the four suspects told the truth, who committed the crime?

(a) Ernesto
(b) Oscar
(c) Paul
(d) Tom

Question 30.

If $F(n+1) = \dfrac{2F(n)+1}{2}$ for $n = 1, 2, \cdots$, and $F(1) = 2$, then $F(101)$ equals:

(a) 49
(b) 50
(c) 51
(d) 52
(e) None of the above.

Question 31.

Given that $\frac{3}{2} < x < \frac{5}{2}$, find the value of

$$\sqrt{x^2 - 2x + 1} + \sqrt{x^2 - 6x + 9}$$

(a) 1
(b) 2
(c) $2x - 4$
(d) $4 - 2x$
(e) None of the above

Question 32.

The square of an integer is called a perfect square. If x is a perfect square, what is the next perfect square?

(a) $x + 1$
(b) $x^2 + 1$
(c) $x + 2x + 1$
(d) $x + 2\sqrt{x} + 1$
(e) None of the above.

Question 33.

Let θ be an acute angle such that $8\cos(2\theta) + 8\sec(2\theta) = 65$. Find the value of $4\cos(4\theta)$.

(a) $-\dfrac{31}{2}$
(b) $-\dfrac{31}{4}$
(c) $-\dfrac{31}{8}$
(d) $-\dfrac{31}{16}$
(e) $-\dfrac{31}{32}$

Question 34.

An odd integer between 600 and 800 is divisible by both 9 and 11. What is the sum of its digits?

(a) 7
(b) 12
(c) 16
(d) 18
(e) 27

Question 35.

How many solutions does the trigonometric equation $\frac{\sin x}{1+\cos x} = 1$ have in the interval $[\pi, 2\pi]$?

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

Question 36.

Find sum of the minimum distance and the maximum distance from the point $(4, -3)$ to the circle $x^2 + y^2 + 4x - 10y - 7 = 0$.

(a) 10
(b) 15
(c) 30
(d) 25
(e) 20

Question 37.

Compare the perimeter of these polygons
(i) A square with side 4 cm
(ii) A rectangle with length 6 cm and width 3 cm
(iii) A triangle with two sides having equal length of 4 cm. The length of the third side is not given.

(a) $(i) < (ii) < (iii)$
(b) $(i) < (iii) < (ii)$
(c) $(iii) < (i) < (ii)$
(d) $(i) = (ii) = (iii)$
(e) None of the above.

Question 38.

For all integers n, $(-1)^{n^4+n+1}$ is equal to

(a) -1
(b) $(-1)^{n+1}$
(c) $(-1)^n$
(d) $(-1)^{n^2}$
(e) 1

Question 39.

Find the minimum value of

$$1 \circ 2 \circ 3 \circ 4 \circ 5 \circ 6 \circ 7 \circ 8 \circ 9$$

where each "$\circ$" is either a "$+$" or a "$\times$".

(a) 36
(b) 40
(c) 44
(d) 45
(e) 84

Question 40.

Given that $a < b$, $x(y - a) = 0$, and $z(y - b) = 0$, which of the following must be true?

(a) $xz < 0$
(b) $xz > 0$
(c) $x = 0$
(d) $z = 0$
(e) $xz = 0$

Answers.

Part I

(1) *b* (2) *c* (3) *c* (4) *b* (5) *b* (6) *c* (7) *c* (8) *d*

(9) *d* (10) *a* (11) *d* (12) *b* (13) *b* (14) *c* (15) *b* (16) *a*

(17) *d* (18) *e* (19) *a* (20) *b* (21) *b* (22) *a* (23) *d* (24) *a*

(25) *c* (26) *c* (27) *c* (28) *d* (29) *c* (30) *b* (31) *e* (32) *a*

(33) *b* (34) *b* (35) *c* (36) *d* (37) *d* (38) *a* (39) *a* (40) *a*

Answers.

Part II

(1) *b* (2) *d* (3) *a* (4) *b* (5) *b* (6) *b* (7) *b* (8) *b*

(9) *a* (10) *c* (11) *b* (12) *c* (13) *e* (14) *a* (15) *c* (16) *c*

(17) *a* (18) *c* (19) *e* (20) *c* (21) *e* (22) *c* (23) *b* (24) *b*

(25) *c* (26) *e* (27) *e* (28) *b* (29) *b* (30) *d* (31) *b* (32) *d*

(33) *c* (34) *d* (35) *a* (36) *e* (37) *c* (38) *a* (39) *c* (40) *e*